MIDDLE EAST TECHNICAL UNIVERSITY

Department of Physics

Formulae Sheet for Qualifying Exam

Constants and Units

Planck's constant	h	$6.64 \times 10^{-34}\,\mathrm{J\,s}$
	$\hbar$	$1.054 \times 10^{-34}\,\mathrm{J\,s}$
Electron mass	m_e	$9.1 \times 10^{-31}\,\mathrm{kg}$
Proton mass	m_p	$1.67 \times 10^{-27}\,\mathrm{kg}$
Electron charge	e	$1.6 \times 10^{-19}\,\mathrm{C}$
Speed of Light	c	$3.0 \times 10^{8}\,\mathrm{m/s}$
Boltzmann's constant	k_B	$1.38 \times 10^{-23}\,\mathrm{J/K}$
Gas constant	R	$8.31\,\mathrm{J/mol{\cdot}K}$
Gravitational constant	G	$6.67 \times 10^{-11}\ \mathrm{N\,m^2/kg^2}$
Permittivity of free space	ϵ_0	$8.85 \times 10^{-12}\,\mathrm{F/m}$
Permeability of free space	μ_0	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Avagadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W/m^2 \cdot K^4}$

T	tera	10^{12}	p	pico	10^{-12}
G	giga	10^{9}	n	nano	10^{-9}
M	mega	10^{6}	μ	micro	10^{-6}
k	kilo	10^{3}	m	mili	10^{-3}

Some Useful Formulae

$dE = đQ - đW = TdS - pdV + \mu dN + \cdots$

Helmholtz Free energy $F = E - TS$, $dF = -SdT - pdV + \mu dN + \cdots$

Gibbs Free energy $G = E - TS + pV$, $dG = -SdT + Vdp + \mu dN + \cdots$

Enthalpy $H = E + pV$, $dH = TdS + Vdp + \mu dN + \cdots$

Fermi-Dirac distribution $n_{FD}(\epsilon) = \dfrac{1}{e^{\beta(\epsilon-\mu)}+1}$,

Bose-Einstein distribution $n_{BE}(\epsilon) = \dfrac{1}{e^{\beta(\epsilon-\mu)}-1}$,

$$\sum_{\vec{\boldsymbol{k}}}(\cdots) = \frac{V}{(2\pi)^3}\iiint d^3\vec{\boldsymbol{k}}(\cdots) ,$$

$$F = -\frac{1}{\beta}\ln Z , \quad Z = \sum_n e^{-\beta E_n} , \quad \beta = \frac{1}{k_B T} ,$$

$$S = k_B\left(-\sum_n p_n \ln p_n\right) , \qquad S = k_B \ln\Omega .$$

$$p_k = \frac{\hbar}{i}\frac{\partial}{\partial x_k} , \qquad [x_\ell, p_k] = i\hbar\delta_{k\ell} ,$$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle , \qquad \frac{d}{dt}\langle A\rangle_t = \langle\frac{\partial A}{\partial t}\rangle_t + \frac{i}{\hbar}\langle[H,A]\rangle_t ,$$

$$A_H(t) = U(t)^\dagger A_S U(t) , \qquad U(t) = \exp\left(-\frac{i}{\hbar}Ht\right) ,$$

Angular momentum $\vec{\boldsymbol{J}}$: $[J_k, J_\ell] = i\hbar\sum_n \epsilon_{k\ell n}J_n$, $J_\pm = J_x \pm iJ_y$,

$$J_z|j,m\rangle = \hbar m|j,m\rangle , \qquad J^2|j,m\rangle = \hbar j(j+1)|j,m\rangle ,$$

$$J_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}|j,m\pm1\rangle ,$$

Spherical coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}}\times\vec{\boldsymbol{p}} , \qquad L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} , \qquad L_\pm = \hbar e^{\pm i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right) ,$$

$$\vec{\boldsymbol{S}} = \frac{\hbar}{2}\vec{\boldsymbol{\sigma}} , \qquad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ,$$

$$\sigma_k\sigma_\ell = \delta_{k\ell}I + i\sum_n \epsilon_{k\ell n}\sigma_n , \qquad (\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{a}})(\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{a}}) = (\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}})I + i(\vec{\boldsymbol{\sigma}}\cdot(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}})) ,$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right)\ ,\quad [a, a^\dagger] = 1\ ,\quad a|n\rangle = \sqrt{n}|n-1\rangle\ ,\quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\ ,$$

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2)\ ,$$

Current density: $\vec{\boldsymbol{J}} = \frac{\hbar}{2mi}\left(\psi^*\vec{\boldsymbol{\nabla}}\psi - c.c.\right),$

Minimal coupling principle: $H = \frac{1}{2m}p^2 \ \longrightarrow\ H = \frac{1}{2m}\left(\vec{\boldsymbol{p}} - \frac{q}{c}\vec{\boldsymbol{A}}\right)^2 + q\phi\ ,$

Gauge transform: $\vec{\boldsymbol{A}} \to \vec{\boldsymbol{A}} + \vec{\boldsymbol{\nabla}}\alpha\ ,\quad \phi \to \phi + \frac{1}{c}\frac{\partial\alpha}{\partial t}\ ,\quad \psi \to \psi e^{-iq\alpha/\hbar}\ ,$

$$H_D = c\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\alpha}} + mc^2\beta\ ,\qquad \left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi = 0\ ,\qquad g_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)\ ,$$

$$\{\alpha_i,\alpha_j\} = 2\delta_{ij}I\ ,\quad \{\alpha_i,\beta\} = 0\ ,\quad \beta^2 = I\ ,\quad \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\ ,$$

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}\ ,\quad \beta = \gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\ ,\quad \gamma^i = \beta\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}\ ,$$

$$E_n^{(1)} = H'_{nn}\ ,\qquad E_n^{(2)} = \sum_{m(\neq n)}\frac{|H'_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}\ ,\qquad |\psi_n^{(1)}\rangle = \sum_{m(\neq n)}|\psi_m^{(0)}\rangle\frac{H'_{mn}}{E_n^{(0)} - E_m^{(0)}}\ ,$$

$$\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!}f^{(n)}(z_0)\,,\quad n = 0,1,2,\cdots\ ,$$

$$\mathrm{Res}\,[f(z),a] = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]\Big|_{z=a}\ ,\qquad \oint_C f(z)dz = 2\pi i\sum_k \mathrm{Res}\,[f(z),a_k]$$

Normalized functions: $\int_a^b U_n^*(\xi)U_m(\xi)d\xi = \delta_{nm}\ ,\qquad \sum_{n=1}^{\infty}U_n^*(\xi')U_n(\xi) = \delta(\xi-\xi')\ ,$

$$\delta(\xi-\xi') = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik(\xi-\xi')}dk,$$

$$\frac{\partial}{\partial z}f(z,z') - \frac{d}{dx}\left(\frac{\partial}{\partial z'}f(z,z')\right) = 0\ ,\qquad \frac{d}{dx}\left(f - z'\frac{\partial}{\partial z'}f(z,z')\right) - \frac{\partial}{\partial x}f(z,z') = 0\ ,$$

$$\int\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)d^3x = \int\left(\phi\vec{\boldsymbol{\nabla}}\psi - \psi\vec{\boldsymbol{\nabla}}\phi\right)\cdot\hat{\boldsymbol{n}}da$$

Gradient operator: $\vec{\boldsymbol{\nabla}} = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$,

Divergence: $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{\Lambda}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\Lambda_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\Lambda_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\Lambda_\varphi}{\partial\varphi}$,

$$\frac{1}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'|} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y^*_{\ell m}(\theta',\varphi')Y_{\ell m}(\theta,\varphi) \ ,$$

$$Y_{10} = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell \ , \quad Y_{00} = \frac{1}{\sqrt{4\pi}} , \quad Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$\int_{-1}^{1}P_\ell(x)P_{\ell'}(x)dx = \frac{2}{2\ell+1}\delta_{ll'} \ , \quad P_0 = 1 \ , \quad P_1 = x = \cos\theta \ ,$$

Potential expansion: $\varphi(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right]P_\ell(\cos\theta)$,

Lorentz transformation:

$$x' = \gamma(x - vt) \ , \quad t' = \gamma(t - vx/c^2) \ , \quad u'_\parallel = \frac{u_\parallel + v}{1 + \frac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{u}}}{c^2}} \ , \quad \vec{\boldsymbol{u}}'_\perp = \frac{\vec{\boldsymbol{u}}_\perp + v}{1 + \frac{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{u}}}{c^2}} \ ,$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \ , \qquad \Lambda^{\mu\nu} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + (\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1 + (\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_y}{\beta^2} & 1 + (\gamma-1)\frac{\beta_z^2}{\beta^2} \end{bmatrix} ,$$

$\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields under LT:

$$\vec{\boldsymbol{E}}' = \gamma\left(\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}\right) - \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}}) \ ,$$

$$\vec{\boldsymbol{B}}' = \gamma\left(\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}\right) - \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{B}}) \ ,$$

Four-vectors: $p^\mu = (E/c, \vec{\boldsymbol{p}})$, $x^\mu = (ct, \vec{\boldsymbol{r}})$, $A^\mu = (\phi/c, \vec{\boldsymbol{A}})$, $J^\mu = (c\rho, \vec{\boldsymbol{J}})$,

Field strength tensor: $F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$,

Poynting vector: $\vec{\boldsymbol{S}} = \vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}$, $\vec{\boldsymbol{g}} = \frac{1}{c^2}\vec{\boldsymbol{S}}$, $\vec{\boldsymbol{J}} = \frac{1}{c^2}\int\vec{\boldsymbol{r}}\times(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}})d^3r$,

Energy-momentum tensor: $T_{\mu\nu} = \epsilon_0\left[E_\mu E_\nu + c^2B_\mu B_\nu - \frac{1}{2}\left(E^2 + c^2B^2\right)\delta_{\mu\nu}\right]$,

Lorentz Force Law: $\frac{dU^\mu}{d\tau} = \frac{e}{mc}F^{\mu\nu}U_\nu$.

Action: $S = \int L(q_j, \dot{q}_j, t)\ dt$, Hamiton's principle: $\delta S = 0$,

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \lambda_k \frac{\partial f_k}{\partial q_j} = 0\ , \quad \frac{d}{dt}\left(L - \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial t}\ ,$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j}\ , \qquad H = p_j \dot{q}_j - L\ , \qquad \langle T \rangle = -\frac{1}{2}\left\langle \sum_{\alpha}^{N} \vec{\boldsymbol{F}}_\alpha \cdot \vec{\boldsymbol{r}}_\alpha \right\rangle\ ,$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j}\ , \quad -\dot{p}_j = \frac{\partial H}{\partial q_j}\ , \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}\ ,$$

Orbit equation: $\frac{\alpha}{r} = 1 + \epsilon \cos\theta$, $\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu^2 r^2}{\ell^2} F(r)$,

$$\left(\frac{d\vec{\boldsymbol{A}}}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{\boldsymbol{A}}}{dt}\right)_{\text{rot}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{A}}\ ,$$

$$I_{ij} = \sum_{\alpha} m_\alpha \left(\delta_{ij} x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right)\ , \qquad L_i = I_{ij}\omega_j\ , \qquad T = \frac{1}{2} I_{ij}\omega_i\omega_j\ ,$$

$$J_{ij} = I_{ij} + M\left(a^2 \delta_{ij} - a_i a_j\right)\ , \qquad \left(I_i - I_j\right)\omega_i\omega_j - \sum_k \left(I_k \dot{w}_k - N_k\right)\epsilon_{ijk} = 0\ ,$$

$$T = \frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j\ , \qquad U = \frac{1}{2}\sum_{ij} V_{ij} q_i q_j\ , \qquad \left|V_{ij} - \omega^2 M_{ij}\right| = 0\ ,$$

$$q_i = P_{ij}\eta_j\ , \qquad P^T M P = 1\ ,$$

Passion bracket : $\{F, G\}_{q,p} = \sum_j \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right)$,

Canonical transformation: $p_i dq_i - H dt = P_i dQ_i - K dt + dF,$

$$F_1 = F_1(q_i, Q_i), \quad F_2 = F_2(q_i, P_i), \quad F_3 = F_3(p_i, Q_i), \quad F_4 = F_4(p_i, P_i)\ ,$$

Hamilton-Jacobi equation: $H\left(q_1, q_2, ..., q_n; \frac{\partial F_2}{\partial q_1}, \frac{\partial F_2}{\partial q_2}, ..., \frac{\partial F_2}{\partial q_n}; t\right) + \frac{\partial F_2}{\partial t} = 0$,