MIDDLE EAST TECHNICAL UNIVERSITY

Sample Exam **Name, Surname :** ____________________

Department of Physics

Time Allowed: 180 Minutes **Signature :**

Instructions:

- This exam contains 6 pages (including the cover page) and 5 problems. Total points is 100.
- All candidates are expected to solve all 5 problems.
- You can use the back of the sheets for solutions.

Problems:	1	2	3	4	5	**Total**
Points:	20	20	20	20	20	100
Score:						

Some Useful Formulas:

- Biot-Savart law

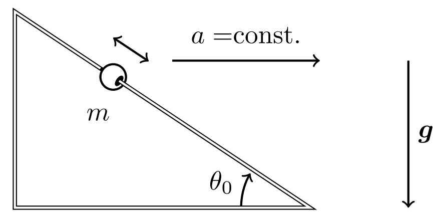
$$d\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{d\boldsymbol{\ell} \times \boldsymbol{r}}{r^3}\,.$$

- Faraday's law

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}\,.$$

- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$
- $H(q,p) = p_j \dot{q}_j - L$
- $t' = \gamma(t - vx/c^2)$
  $x' = \gamma(x - vt)$
  $y' = y$
  $z' = z$
  $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$

1. A massless wire is bent into the shape of a right triangle whose diagonal makes an angle θ_0 with the horizontal. The wire in vertical position is put in motion along a horizontal line with a constant acceleration $\boldsymbol{a}$. There is a bead of mass m on the diagonal part of the wire. At the initial time $t = 0$, the bead is positioned somewhere on the wire and left at rest relative to the wire. There is a uniform gravity pointing downward. The snapshot of the motion at any time t is given in the figure. Ignore any frictional forces.

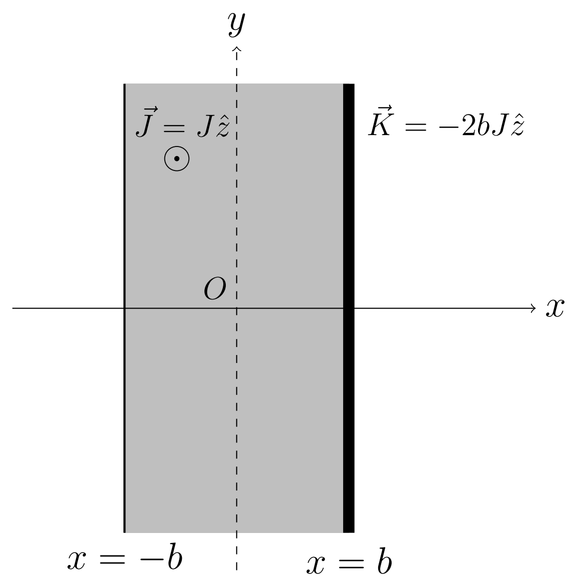


$\boxed{10}$ (a) Using the distance of the bead from the top point as a generalized coordinate, write down the Lagrangian of the system.
Hint: Use an inertial frame of reference to express the Lagrangian.

$\boxed{4}$ (b) Obtain the Euler-Lagrange equation of motion.
What should be the acceleration of the wire in terms of the given parameters in order for the bead to remain where it is initially?

$\boxed{6}$ (c) Without constructing the Hamiltonain H of the system explicitly, explain the following clearly:
   i. Is H constant in time?
   ii. Is H equal to the total energy, E?
   iii. Is E constant in time?

2. A thick slab extending from $x = -b$ to $x = b$ and infinite in the y and z directions is carrying a volume current density given as $\boldsymbol{J} = J\hat{\boldsymbol{z}}$. There is also a surface current density $\boldsymbol{K} = -2Jb\hat{\boldsymbol{z}}$ at the planar interface at $x = b$.



$\boxed{6}$ (a) Using Ampere's law, determine the magnetic field $\boldsymbol{B}_{slab}$ due to the slab only as a function of the coordinate x both inside and outside the slab. What is the magnetic field along the y-axis?
Hint: Choose rectangular Amperian loops on the $x - y$ plane, whose center is on the x-axis. Decide on the direction of the magnetic field in each region using the right hand rule.

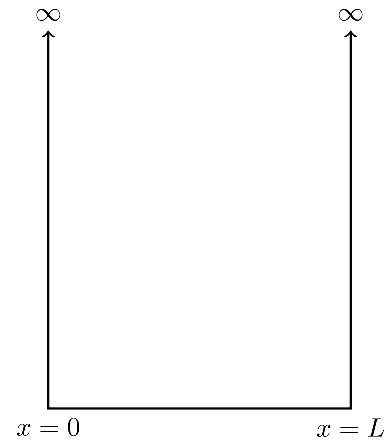
$\boxed{5}$ (b) Determine the magnetic field $\boldsymbol{B}_{sheet}$ due to the infinite planar interface at $x = b$ only, in all regions both inside and outside the slab.
Hint: Choose a rectangular Amperian loop on the $x - y$ plane whose center is on the plane $x = b$. Use the right hand rule to figure out the direction of the magnetic field on either side of the interface.

$\boxed{4}$ (c) Using the principle of superposition, find the total magnetic field $\boldsymbol{B}$ in all regions both inside and outside slab.

$\boxed{5}$ (d) Consider a square region of side length a on the $x - z$ plane centered at the origin of the coordinates inside the slab. Compute the flux of the total magnetic field through this region.

3. A particle is trapped between infinite potential walls located at $x = 0$ and $x = L$. It has the initial state wave function composed of the first two normalized eigenstates,

$$\Psi(x\,,0) = A\big[\psi_1(x) + \psi_3(x)\big]\ .$$



$\boxed{6}$ (a) Solve the time-independent Schrödinger equation and explicitly write down the normalized eigenstates $\psi_1(x)$ and $\psi_3(x)$.

$\boxed{4}$ (b) Determine the normalization constant A.

$\boxed{5}$ (c) Explicitly write down $\Psi(x, t)$.

$\boxed{5}$ (d) Compute expectation value of the energy at time t. Is it conserved? Explain why or why not.

4. Consider the canonical ensemble for a system at temperature T.

$\boxed{5}$ (a) Show that the average energy in the canonical ensemble can be computed as

$$\overline{E} = -\frac{\partial \ln Z}{\partial \beta} \ ,$$

where Z is the partition function and $\beta = 1/k_B T$.

$\boxed{5}$ (b) Consider a harmonic oscillator with frequency ω. If we drop the zero-point energy, the energy levels are given by

$$E_n = n\hbar\omega \ , \qquad (n = 0, 1, 2, 3, \ldots).$$

Find the partition function Z for this oscillator.

$\boxed{5}$ (c) Find the average energy $\overline{E}$ of the oscillator.

$\boxed{5}$ (d) Show that, at high temperatures, the average energy is approximately given by the equipartition formula (i.e., as $T \to \infty$, $\overline{E} \approx k_B T$).

5. Answer the following questions.

$\boxed{5}$ (a) Find the solution of the equation $y' + 2y = 6$ with the initial condition $y(0) = 4$.

$\boxed{5}$ (b) A positronium consists of an electron and a positron in a state of spin angular momentum $s = 1$ and total angular momentum $j = 1$. What are the possible values of the orbital angular momentum quantum number ℓ for the positronium?

$\boxed{5}$ (c) The surface waves on water in deep seas have the dispersion relation (ω vs k relation) given by $\omega = \sqrt{gk}$ where g is a constant. Consider a "wave packet" of water waves.

i. Find the phase velocity, v_{ph}.
ii. Find the group velocity, v_{gr}.
iii. Compute the ratio v_{ph}/v_{gr}.

$\boxed{5}$ (d) Assume that there are two clocks which are a distance d apart on the x' axis of a K' frame. They are both at rest in K' frame. We are at the origin of a stationary frame K and the K' frame moves with a relativistic velocity v along the x' direction with respect to us. The x and x' axes are parallel to each other. We observe that the clocks are synchronized, in other words, each tick of the clocks happen <u>at the same time</u> relative to us. How much are they out of synchronization in their rest frame (that is, in the K' frame)? Express it in terms of the given parameters.