Sample Exam
Department of Physics

Time Allowed: 180 Minutes

Instructions:

• This exam contains 6 pages (including the cover page) and 5 problems. Total points is 100.
• All candidates are expected to solve all 5 problems.
• You can use the back of the sheets for solutions.

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Score:

Some Useful Formulas:

• Biot-Savart law
  \[ dB = \frac{\mu_0}{4\pi} \frac{d\ell \times r}{r^3} . \]

• Faraday’s law
  \[ \oint E \cdot d\ell = -\frac{d\Phi_B}{dt} . \]

• \[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \]

• \( H(q, p) = p_j \dot{q}_j - L \)

• \( t' = \gamma(t - vx/c^2) \)
  \( x' = \gamma(x - vt) \)
  \( y' = y \)
  \( z' = z \)
  \( \gamma \equiv 1/\sqrt{1 - v^2/c^2} \)
1. A massless wire is bent into the shape of a right triangle whose diagonal makes an angle $\theta_0$ with the horizontal. The wire in vertical position is put in motion along a horizontal line with a constant acceleration $a$. There is a bead of mass $m$ on the diagonal part of the wire. At the initial time $t = 0$, the bead is positioned somewhere on the wire and left at rest relative to the wire. There is a uniform gravity pointing downward. The snapshot of the motion at any time $t$ is given in the figure. Ignore any frictional forces.

(a) Using the distance of the bead from the top point as a generalized coordinate, write down the Lagrangian of the system. 
   Hint: Use an inertial frame of reference to express the Lagrangian.

(b) Obtain the Euler-Lagrange equation of motion. 
   What should be the acceleration of the wire in terms of the given parameters in order for the bead to remain where it is initially?

(c) Without constructing the Hamiltonian $H$ of the system explicitly, explain the following clearly:
   i. Is $H$ constant in time?
   ii. Is $H$ equal to the total energy, $E$?
   iii. Is $E$ constant in time?
2. A thick slab extending from \( x = -b \) to \( x = b \) and infinite in the \( y \) and \( z \) directions is carrying a volume current density given as \( \mathbf{J} = J\hat{z} \). There is also a surface current density \( \mathbf{K} = -2Jb\hat{z} \) at the planar interface at \( x = b \).

(a) Using Ampere’s law, determine the magnetic field \( \mathbf{B}_{\text{slab}} \) due to the slab only as a function of the coordinate \( x \) both inside and outside the slab. What is the magnetic field along the \( y \)-axis?
   
   Hint: Choose rectangular Amperian loops on the \( x-y \) plane, whose center is on the \( x \)-axis. Decide on the direction of the magnetic field in each region using the right hand rule.

(b) Determine the magnetic field \( \mathbf{B}_{\text{sheet}} \) due to the infinite planar interface at \( x = b \) only, in all regions both inside and outside the slab.

   Hint: Choose a rectangular Amperian loop on the \( x-y \) plane whose center is on the plane \( x = b \). Use the right hand rule to figure out the direction of the magnetic field on either side of the interface.

(c) Using the principle of superposition, find the total magnetic field \( \mathbf{B} \) in all regions both inside and outside slab.

(d) Consider a square region of side length \( a \) on the \( x-z \) plane centered at the origin of the coordinates inside the slab. Compute the flux of the total magnetic field through this region.
3. A particle is trapped between infinite potential walls located at \( x = 0 \) and \( x = L \). It has the initial state wave function composed of the first two normalized eigenstates,

\[
\Psi(x, 0) = A[\psi_1(x) + \psi_3(x)] .
\]

(a) Solve the time-independent Schrödinger equation and explicitly write down the normalized eigenstates \( \psi_1(x) \) and \( \psi_3(x) \).

(b) Determine the normalization constant \( A \).

(c) Explicitly write down \( \Psi(x, t) \).

(d) Compute expectation value of the energy at time \( t \). Is it conserved? Explain why or why not.
4. Consider the canonical ensemble for a system at temperature $T$.

(a) Show that the average energy in the canonical ensemble can be computed as

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta} ,$$

where $Z$ is the partition function and $\beta = 1/k_B T$.

(b) Consider a harmonic oscillator with frequency $\omega$. If we drop the zero-point energy, the energy levels are given by

$$E_n = n\hbar \omega , \quad (n = 0, 1, 2, 3, \ldots).$$

Find the partition function $Z$ for this oscillator.

(c) Find the average energy $\bar{E}$ of the oscillator.

(d) Show that, at high temperatures, the average energy is approximately given by the equipartition formula (i.e., as $T \to \infty$, $\bar{E} \approx k_B T$).
5. Answer the following questions.

(a) Find the solution of the equation $y' + 2y = 6$ with the initial condition $y(0) = 4$.

(b) A positronium consists of an electron and a positron in a state of spin angular momentum $s = 1$ and total angular momentum $j = 1$. What are the possible values of the orbital angular momentum quantum number $\ell$ for the positronium?

(c) The surface waves on water in deep seas have the dispersion relation ($\omega$ vs $k$ relation) given by $\omega = \sqrt{gk}$ where $g$ is a constant. Consider a “wave packet” of water waves.
   i. Find the phase velocity, $v_{ph}$.
   ii. Find the group velocity, $v_{gr}$.
   iii. Compute the ratio $v_{ph}/v_{gr}$.

(d) Assume that there are two clocks which are a distance $d$ apart on the $x'$ axis of a $K'$ frame. They are both at rest in $K'$ frame. We are at the origin of a stationary frame $K$ and the $K'$ frame moves with a relativistic velocity $v$ along the $x'$ direction with respect to us. The $x$ and $x'$ axes are parallel to each other. We observe that the clocks are synchronized, in other words, each tick of the clocks happen at the same time relative to us. How much are they out of synchronization in their rest frame (that is, in the $K'$ frame)? Express it in terms of the given parameters.