Department of Physics
Qualifying Exam Questions
(Without Miscellaneous Questions)

Statistical Mechanics


(a) Unlike the ideal gas that is assumed to be made up of noninteracting point particles, real gas atoms/molecules have a finite volume and interact. A more realistic description of a gas is the van der Waals model. In this model, the equation of state and the internal energy are

\[
\left[ P + a \left( \frac{n}{V} \right)^2 \right] \left( \frac{V}{n} - b \right) = RT, \\
U = \frac{3}{2} nRT - \frac{n^2 a}{V}
\]

where \( P, V \) and \( n \) are the pressure, volume and number of moles, respectively. The constants \( a \) and \( b \) are positive values that model finite size of atoms and finite inter-particle interaction. Calculate the entropy change during an isothermal process that occurs at \( T = T_0 \) where the final volume is twice the initial volume, \( V_0 \).

(b) Two identical blocks of copper, one at \( T_1 \) and the other at \( T_2 \) are placed in thermal contact with each and are thermally isolated from everything else. Given that the heat capacity at constant volume of each block, \( C \), is independent of temperature, obtain an expression for the change in entropy when the system reaches equilibrium with respect to the initial state. Also, show that \( \Delta S > 0 \) regardless of whether \( T_1 > T_2 \) or \( T_2 > T_1 \). (Assume \( T_1 \neq T_2 \).)

*Hint: The arithmetic-geometric mean inequality states that the geometric mean is always smaller than the arithmetic mean: \( \sqrt{xy} \leq (x + y)/2 \).*

(c) A one-dimensional quantum harmonic oscillator (whose ground state energy is \( \hbar \nu \)) is in thermal equilibrium with a heat bath at temperature \( T \). What is the mean value of the oscillator’s energy \( \langle E \rangle \) as a function of \( T \)?

(d) Consider a system of \( N \) non-interacting particles \( (N \gg 1) \) in which the energy of each particle can assume only two values, \( \varepsilon_0 = 0 \) and \( \varepsilon_1 = E \) where \( (E > 0) \). If in a particular macrostate, the occupation number of the state with label \( \varepsilon_0 \) is \( n_0 \), find the temperature, \( T \), of the system as a function of the total internal energy \( U \).
(e) In our three-dimensional universe, the energy density of black body radiation depends on the temperature as \( T^\alpha \) where \( \alpha = 4 \). What is the value of \( \alpha \) in an \( n \)-dimensional universe?

Hint: You will end up with a complicated integral over \( \omega \). You can extract the temperature dependence without actually evaluating the integral.

[SM-2017-May] Q2: Dipolar molecules

An ideal classical gas is formed by \( N \) indistinguishable non-interacting diatomic molecules. Each one of the has an electric dipole moment of magnitude \( D \). The whole gas is in thermal equilibrium at temperature \( T \) and is under the effect of a constant electric field with intensity \( \mathcal{E} \) directed along the \( z \) axis. The Hamiltonian of a single dipole is

\[
\mathcal{H} = \frac{1}{2I} p_\theta^2 + \frac{1}{2I \sin^2 \theta} p_\phi^2 - DE \cos \theta
\]

where \( I \) is the moment of inertia of the molecule, \((\theta, \phi)\) are the spherical polar angles to represent the orientation and \((p_\theta, p_\phi)\) are the associated momenta. The Hamiltonian therefore contains contributions from the rotational degrees of freedom of the molecules as well as the coupling of the dipoles with the electric field.

(a) Sketch the vectors \( \vec{D} \) and \( \vec{E} \) on the given axis in the figure and mark the angles clearly.

(b) In the analytical mechanics exam that you took yesterday, you found out how the phase space volume element \( dv \) changes under a canonical coordinate transform. For this question, in particular, going from Cartesian to spherical coordinates in phase space

\[
(x, y, z, p_x, p_y, p_z) \rightarrow (r, \theta, \phi, p_r, p_\theta, p_\phi)
\]

causes the volume element to transform as

\[
dv = dx dy dz dp_x dp_y dp_z \rightarrow dv = dr d\theta d\phi dp_r dp_\theta dp_\phi
\]

and since, in a rotational Hamiltonian, the radius does not change, you only need

\[
dv = d\theta d\phi dp_\theta dp_\phi.
\]

Using this information, write down the classical partition function for a single particle. Do not perform any integrations at this stage.

(c) Perform the integrations in the previous part to prove that the partition function of a single dipole is given by

\[
Q_1 = \frac{2I \sinh(\beta D \mathcal{E})}{\hbar^2 \beta^2 D \mathcal{E}}
\]

Hint: \( \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}. \)

(d) Write down the \( N \)-particle partition function, \( Q_N \).

(e) Calculate the specific heat, \( C_V \).

(f) Calculate the high temperature limit of \( C_V \).

(g) When \( \mathcal{E} \rightarrow 0 \), the system reduces to a regular collection of diatomic molecules. In this limit, does the specific heat yield the result you would expect from the equipartition theorem? Explain briefly.
Macroscopic polarization is defined as
\[ P = \frac{N}{V} (D \cos \theta). \]

Starting from \( Q_N \) derived above, show that
\[ P = \frac{N}{V} \left( D \coth(\beta D \mathcal{E}) - \frac{1}{\beta \mathcal{E}} \right) \]

In the limit of weak field \( \beta D \mathcal{E} \to 0 \), show that the dielectric constant defined by
\[ \epsilon \mathcal{E} = \epsilon_0 \mathcal{E} + P \]
is equal to \( \epsilon = \epsilon_0 + \frac{N \beta D^2}{3V} \).

**[SM-2016-Nov] Q1: Answer the Following Questions.**

*Note: The individual parts of the following question are intended to be independent from each other.*

(a) For most of the gases at room temperature, the vibrational modes are frozen and the translational and rotational degrees of freedom can be treated classically. Using the equipartition theorem, determine the molar heat capacity of
(i) Ne gas,
(ii) CO\(_2\) gas (note that CO\(_2\) molecules are linear), and
(iii) H\(_2\)O gas.

(b) Consider a photon gas inside a cavity of volume \( V \) and temperature \( T \). The volume of the cavity is then expanded to volume \( 2V \) in such a way that (1) the expansion is so quick that there is no appreciable energy (heat) transfer from the walls of the cavity to the photon gas and (2) the walls move at non-relativistic speeds so that the change is “slow” for the photon gas.
(i) Discuss if this is an example of an adiabatic process for the photon gas.
(ii) Find the final temperature of the gas.
*Hint: The Helmholtz free energy of the photon gas is given as \( F(T,V) = -b T^4 V \) where \( b \) is some constant.*

(c) \( N \) independent and distinguishable point particles move in a one-dimensional domain between \( q = 0 \) and \( q = L \). Determine the one-dimensional pressure in the whole system, if the single-particle Hamiltonian is given by
\[ \mathcal{H} = \mathcal{H}(p,q) = \frac{p^2}{2m} - \alpha \ln \left( \frac{q}{L_0} \right), \quad (\alpha > 0). \]
Assume that the system is in a canonical ensemble. In the above expression, \( \alpha \) is a constant giving the strength of the potential and \( L_0 \) is a characteristic length scale. What is the low-temperature limit of the pressure?

(d) In a Fermi gas of \( N \) spin \( s = 1/2 \) particles with mass \( m \), the particles occupy a two-dimensional domain with an area \( A \). If the temperature is \( T \), determine the Fermi energy, \( \varepsilon_F \), as a function of particle density, \( n \). (Assume \( k_B T \ll \varepsilon_F \).)

(e) The Joule expansion of a gas is the sudden expansion of the gas from an initial volume \( V_i \) to a final volume \( V_f \). The expansion is so fast that there is not enough time for heat absorption (as a result it can be assumed that the gas is thermally insulated). After that, the gas is allowed to equilibrate. Joule’s empirical work led to the observation that the temperature \( T \) of the gas does not change during and after the expansion process.
Assume that the gas obeys the ideal gas law. Answer the following questions:
(i) Give a plausible explanation for the fact that the temperature doesn’t change.
(ii) Why is this an irreversible process?
(iii) Can the entropy change $\Delta S$ between the initial and final states be calculated using a process like this?
(iv) If $V_f = 2V_i$, calculate the entropy change during the expansion.

[SM-2016-Nov] Q2: Hanging Chain

A one dimensional chain, made of massless rings, hangs from a ceiling. One of its extremes is fixed, while the other holds a mass $M$ as shown in figure. Gravity acts along the negative $z$ direction. The chain is formed by two kinds of rings: they are ellipses with the major axis oriented vertically or horizontally. The major and minor axes have lengths $(l + a)$ and $(l - a)$, respectively. Although the number of the rings is fixed, the rings are allowed to change orientation (vertical to horizontal) in accordance with the finite temperature. The number of rings in the vertical direction for a given state of the chain is $n$.

(a) Write down the total length, $L$, of the chain in terms of $n$ and the other relevant constants.
(b) What is the internal energy of the chain for a given $n$?
(c) For a given length $L$, determine the number of possible microstates, $g(n)$. Explain your answer briefly.
(d) Using $g_n$ and assuming a canonical distribution, write down and simplify the partition function, $Q_N$, for the entire chain. Note that the rings are distinguishable.
   Hint 1: You will have to assign energies to the vertical and horizontal orientations.
   Hint 2: Depending on your approach, you may or may not need the binomial expansion,
   $$(x + y)^M = \sum_k \binom{M}{k} x^k y^{M-k}.$$ 
(e) Using $Q_N$ as the starting point, derive the entropy of the system. Find the high and low temperature limits.
(f) Calculate the average length, $\langle L \rangle$. Find the limit of the length at the high and low temperature limits.
(g) Show that the linear response at the high temperature limit, which is defined as the change in the average length as a function of the force $F = Mg$,
   $$\chi = \frac{\partial \langle L \rangle}{\partial F} = \frac{Na^2}{kT}.$$ 
(h) This result formally resembles the magnetic susceptibility
   $$\chi = \frac{\partial \langle M \rangle}{\partial H} = \frac{N\mu^2}{kT}$$
   of a one-dimension system of up and down spins under an external B-field. This is the well-known Curie law and $M$ is the total magnetization of the system. Explain the resemblance by drawing analogies to $\mu$ and $H$ in the current problem of the hanging chain.

[SM-2016-May] Q1:
Answer the Following Questions.
(a) In a canonical ensemble, the average of the thermal fluctuations in the internal energy can be calculated using
\[ \langle \Delta E^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2. \]
Prove that \( \langle \Delta E^2 \rangle = k_B T^2 C_V. \)

(b) A system has three single-particle energy states, two of which are degenerate: \( E_1 = E_2 = \varepsilon, E_3 = 2\varepsilon. \) Write down three canonical partition functions for a two-particle system if the particles are
(i) distinguishable,
(ii) fermions,
(iii) bosons.

(c) A body of constant heat capacity, \( C_P \) and at a temperature \( T_i \) is put in contact with a heat reservoir, which is at temperature \( T_f. \) Equilibrium between the body and the reservoir is established at constant pressure. Determine the total entropy change, \( \Delta S \) of the body and the reservoir and prove that it is positive for both \( T_i > T_f \) and \( T_f > T_i. \) (Hint: Consider \( \Delta S \) as a function of \( x = T_f/T_i \) and compute its minimum).

(d) The Helmholtz free energy of a photon gas in a cavity of volume \( V \) is
\[ F = -\frac{\pi^2 k_B^4 T^4 V}{45\hbar^3 c^3}. \]
(i) Find the entropy, \( S. \)
(ii) Find the internal energy, \( U.\)
(iii) Find the pressure, \( P, \) and briefly comment on why the pressure is nonzero in a system of particles with no mass.
(iv) Write down the equation of state, i.e., write down \( U \) as a function of \( P \) and \( V.\)

(e) Transverse waves on the surface of liquid He (ripplons) have a dispersion relation given by \( \omega(k) = (\gamma_s k^3/\rho)^{1/2}, \) where \( k \) is the norm of the wave vector associated with a given wave, \( \gamma_s \) is the surface tension and \( \rho \) is the mass density. The two-dimensional density of states is \( D(k) = \frac{A}{\pi} k, \) where \( A \) is the area. Show that the average internal energy per unit area for the ripplons has the form \( cT^n, \) where \( c \) is a constant, and find \( n. \)
Hint:
1. Ripplons obey the Bose-Einstein distribution.
2. Note that \( D(k)dk \) is the number of states whose k-vector has magnitude between \( k \) and \( k + dk. \)

**[SM-2016-May] Q2: Adiabatic Demagnetization**

Consider a paramagnetic substance inside a magnetic field. If the strength of the magnetic field is reduced adiabatically (i.e., slowly with no heat exchange with the outside), the temperature of the substance drops. This phenomenon is frequently used as a refrigeration technique. As it requires no moving parts, refrigeration by adiabatic demagnetization is usually preferred to traditional cooling techniques, especially at very low temperatures. It has been used successfully to achieve extremely low temperatures. In this problem, we will see the physical basis of this effect.

Consider a substance containing \( N \) atoms with spin 1/2. When an external magnetic field \( B \) along the \( z \) direction is applied, the magnetic energy of the substance is given by
\[ E^{mag} = \sum_{i=1}^{N} -\mu_{i,z} B \]
where, \( \mu_{i,z}, \) the magnetic moment of the \( i \)th atom, takes on two possible values: \( \mu_{i,z} = \pm \mu_0. \)
(a) Assuming canonical ensemble at temperature $T$, compute the entropy $S^{mag}$ of the substance. Express it as a function of the dimensionless parameter $x = \mu_0 B/k_B T$. Find the two limiting values of entropy (i) for $x \to 0$ and (ii) for $x \to \infty$.

(b) Explain the dependence of the entropy on the two important parameters, $T$ and $B$, by answering the following.
(i) Sketch the $S^{mag}$ vs $T$ graph. Does entropy increase or decrease with $T$? What is the physical reason for this dependence?
(ii) Sketch the $S^{mag}$ vs $B$ graph. Does entropy increase or decrease with $B$? What is the physical reason for this dependence?

(c) Which quantity remains invariant in an adiabatic process? Why?

(d) Suppose that initially the substance has temperature $T_i$ and field $B_i$. The magnetic field is then reduced adiabatically to half of its initial value, $B_f = B_i/2$. What is the final temperature, $T_f$, of the substance?

(e) Suppose that the magnetic field is completely removed, i.e. $B_f = 0$. What should be the final temperature in this case? Is this physically possible?

In a realistic substance, the translational motion of the atoms makes a contribution to the total entropy. When computing $T_f$ in an adiabatic demagnetization process, this contribution should also be taken into account. Suppose that the substance we are considering is a monatomic gas. In that case, the total entropy is $S = S^{gas} + S^{mag}$ where $S^{gas}$ is the entropy due to the translational motion of the atoms.

(f) Compute $S^{gas}$ as a function of $T$.

(g) Consider a demagnetization process where the initial magnetic field $B_i$ is very high ($x_i \gg 1$) and the final magnetic field is $B_f = 0$. Compute the final temperature $T_f$ of the substance by taking into account both contributions to the entropy.

[SM-2015-Nov] Q1:
Answer the Following Questions.

(a) The heat capacity (at constant volume) of a solid material at low temperature $T$ is given by $C = aT^3$ where $a$ is some fixed constant.
(i) Find entropy $S$ as a function of temperature.
(ii) Find the internal energy $E$ as a function of temperature.
Note: ignore all volume dependencies in this problem and treat $S$ and $E$ as functions of $T$ only.

(b) Consider a photon gas at temperature $T$ in volume $V$. Write down the expression for the total energy $E$ as a frequency integral. After that, without evaluating the integral, show that $E \propto T^n$ for some power $n$ and determine $n$.

(c) A refrigerator engine does the following in one cycle of its operation: First, it absorbs 25 J heat from the freezer which is at -23°C temperature. Then, it dumps 45 J of heat to the outside room which is at temperature 27°C.
(i) How much electrical energy did the engine use over one cycle?
(ii) Computing the changes in entropy of freezer, room and the engine, decide if this kind of an engine is physically possible.

(d) Consider an atom with three energy levels with energies $E_1 = 0$ and $E_2 = E_3 = \varepsilon$. Find the average energy $\langle E \rangle$ and entropy $S$.
(i) ... at the zero-point, $T = 0$.
(ii) ... at $T = \infty$.

(e) For an ideal gas of $N$ molecules inside volume $V$ having total energy $E$, the number of microstates is given as
$$\Omega = \Omega(E,V,N) = c_N E^{7N/2} V^N,$$
where $c_N$ depends only on $N$. Use this expression to answer the following.
(i) Find the temperature $T$ and express $E$ vs $T$ relation.
(ii) Find the pressure $p$.

(f) Consider a one-dimensional metal with a separation $a$ between atoms. Assume that there is one electron of mass $m$ in each atom. Calculate the Fermi energy at $T = 0$ in terms of the given parameters.


Paramagnetism of matter arise from the electron spin. Consider a free electron gas, for example those in a metal. If a magnetic field $B$ is applied, each electron will have an energy given by

$$\epsilon_i = \frac{p_i^2}{2m} + \sigma_i \mu_B B,$$

where $\mu_B$ is the Bohr magneton, $\vec{p}_i$ is the momentum of the $i$th electron and $\sigma_i = \pm 1$ depending on whether the spin of the $i$th electron is parallel or antiparallel to the applied field. We treat this as a degenerate Fermi gas at Fermi energy $\epsilon_F$, i.e., take $T = 0$.

(a) Since the two spin orientations have different energies, the Fermi wavenumber $k^\sigma_F$ depends on the spin orientation $\sigma$. Find $k^+_F$, Fermi momentum for spin-up electrons, and $k^-_F$, Fermi momentum for spin-down electrons.

(b) Let $V$ be the volume occupied by the electrons. Find $N^+$, the total number of spin-up electrons, and $N^-$, the total number of spin-down electrons. Express $N = N^+ + N^-$, as a function of $\epsilon_F$ and $V$.

(c) The total magnetic moment of the electron gas in the direction of applied field can be expressed as

$$\mu_{tot} = \sum_i (-\mu_B) \sigma_i = (N^- - N^+) \mu_B,$$

and the magnetization density is $M = \mu_{tot}/V$. Compute $M$.

(d) Find the susceptibility of the electron gas

$$\chi = \left( \frac{\partial M}{\partial B} \right)_{B=0}.$$

(e) In metallic copper, the electron number density is

$$n = \frac{N}{V} \sim 8 \times 10^{28} \text{ m}^{-3}.$$

Compute the numerical value of the parametric contribution to susceptibility $\chi$. (Take $\mu_B = 9.27 \times 10^{-24} \text{ J/T}$).

(f) Consider a hypothetical situation where the volume of the gas is halved $V \rightarrow V/2$ while keeping the number of electrons $N$ fixed. By which ratio do the Fermi energy $\epsilon_F$ and the paramagnetic susceptibility change?

(a) For a photon gas at temperature $T$ occupying a volume $V$, the total energy is given by the Stefan-Boltzmann law $E = aVT^4$ where $a$ is a universal constant. Use this, and the 1st law of thermodynamics

$$dE = TdS - pdV$$

to compute the entropy $S$, and the pressure $p$ of the photon gas as a function of $T$ and $V$. 

*Hint: Consider $S$ and $p$ as functions of $V$. Which one is extensive, which one is intensive?*

(b) The Helmholtz free energy of a gas of $N$ molecules at temperature $T$, occupying a volume $V$ is given as

$$F = F(T, V, N) = -k_BNT \left( \ln \frac{VT^{5/2}}{N} + c_0 \right)$$

where $c_0$ is some constant. Find the entropy $S$, the internal energy $E$ and the pressure $p$ of the gas (as a function of $T$, $V$, $N$).

(c) In an experiment investigating matter at $T_c = 1K$, a refrigerator is used for maintaining this temperature. The refrigerator works as follows: It uses electrical energy $W$ to absorb heat $Q_c$ from the cold matter at $T_c$ and dumps the heat $Q_h$ to the environment at room temperature at $T_h = 300K$. If $Q_c = 2J$ of heat is wanted to be removed, find the minimum possible value of the waste heat $Q_h$ and the minimum possible energy $W$ to be used for achieving this task.

(d) Let $n = n(\epsilon, \mu, T)$ be the average number of particles in an orbital with energy $\epsilon$ when the particles have chemical potential $\mu$ at temperature $T$. Carefully sketch a plot of $n$ vs $\epsilon$ graph for the cases

i. particles are bosons,
ii. particles are fermions.

Indicate the location of the chemical potential $\mu$ on the graphs.

(e) Describe Gibbs’ paradox. Why and how does it appear? How do we solve it?

(f) For a hypothetical system, the number of microstates at energy $E$ is given by

$$\Omega(E) = E^a e^{bE}$$

where $a$ and $b$ are constants. Find energy $E$ as a function of temperature $T$.

(g) Consider a 3-level atom. Suppose that the levels have energies $-\epsilon$, 0 and $2\epsilon$ ($\epsilon > 0$).

i. What is the probability $p$ that the atom is in the ground state at temperature $T$?

ii. What is the limiting value of $p$ at $T \rightarrow 0$ limit?

iii. What is the average energy of the atom at $T \rightarrow 0$ limit?

iv. What is the average energy of the atom at $T \rightarrow \infty$ limit?

[SM-2015-May] Q2: Statistical mechanics of a harmonic oscillator

Consider a harmonic oscillator with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2,$$

in equilibrium with an environment at temperature $T$. We may use either “classical statistical mechanics” or “quantum statistical mechanics” to describe the oscillator and, depending on the value of the temperature, there might be differences between these two approaches. In this problem, we will compare these two approaches.

(a) Before we proceed, let us first derive a general formula. Show that, for an arbitrary system in canonical equilibrium, the average energy can be computed as $E = -\partial \ln Z/\partial \beta$, where $Z$ is the canonical partition function and $\beta = 1/k_BT$.

**Part I:** Let us start with the classical statistical mechanics of the oscillator.
(b) Compute the canonical partition function $Z_{cl}$ (as a phase-space integral).

(c) Use $Z_{cl}$ to find the average energy $E_{cl}$ of the oscillator as a function of $T$.

(d) Describe equipartition theorem. Is $E_{cl}$ consistent with this?

**Part II:** Now, we will do the quantum statistical mechanics.

(e) What are the exact energy eigenvalues $E_n$ of the oscillator? Use these to compute the canonical partition function $Z_q$. (Note: You have to include the zero-point energy part of $E_n$ into your calculations!)

(f) Use $Z_q$ to find the average energy $E_q$ of the oscillator.

(g) To be able to compare $E_q$ and $E_{cl}$ look at the behavior at the following limits.
   
   i. High-temperature limit, $k_B T \gg \hbar \omega$ (or $\beta \hbar \omega \ll 1$): Expand $E_q$ as a power series in $T$,
      
      \[ E_q = k_B T + a_0 + \frac{a_1}{T} + \frac{a_2}{T^2} + \cdots. \]
      
      Show that $a_0 = 0$ and compute $a_1$.
      
      *Hint: First expand $1/(e^x - 1)$ for small $x$ up to at least the 1st order.*

   ii. Low-temperature limit, $k_B T \ll \hbar \omega$ (or $\beta \hbar \omega \gg 1$): What is the dominant dependence of $E_q$ on $T$?
   
   iii. On the same graph, sketch both $E_q$ and $E_{cl}$ as a function of $T$. Ensure that both high and low temperature behaviors are correctly sketched.

(h) Based on this problem what can you say about which approach (classical or quantum) should/could be used under which conditions?

(i) Molecular vibrations have frequencies of the order of magnitude $\omega \sim 10^{14} \text{rad/s}$. Based on the analysis above, what can you say about which approach should be used around the room temperature $T \sim 300 \text{K}$ for describing molecular vibrations?

*Hint:*

\[ \int_{-\infty}^{+\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}}. \]
Quantum Mechanics


Note: The individual parts of the following question are intended to be independent from each other.

(a) The hyperfine interaction is the name given to the magnetostatic interaction of the internal magnetic moments of the electron and the proton in the hydrogen atom. For the case when the electron is in the 1s level, the spin states of the proton and electron are governed by the Hamiltonian

\[ H_{hf} = A \vec{S}_e \cdot \vec{S}_p , \]

where \( A \) is a positive constant. Consider only the spins states.

(i) Does this Hamiltonian has rotational symmetry? (Explain in words, i.e., do not try to compute a commutator.) If so, which observable is conserved? (Words only.)

(ii) Find all energy eigenstates of \( H_{hf} \). Find also the energy eigenvalues and their degeneracies. Which level is the ground state?

(iii) Find the frequency of photons emitted due to transitions between these levels.

Note: this is the famous 21 cm line that comes from atomic hydrogen in interstellar nebulae.

(b) For a harmonic oscillator in 1D, carefully sketch

(i) the wavefunctions and

(ii) the probability density

as a function of position for the lowest three levels.

(c) The Dirac Hamiltonian is given by \( H_D = c \vec{\alpha} \cdot \vec{p} + mc^2 \beta \).

(i) Compute the time derivative

\[ \frac{d}{dt} \langle x_i \rangle \]

for an arbitrary state. Using this, identify the velocity operator.

(ii) Is it possible to measure two different components of the velocity at the same time?

(iii) What are the eigenvalues of the \( x \)-component of the velocity operator?

(d) Consider the following Hamiltonian

\[ H = \frac{1}{2m} \left( \vec{p} + \frac{eB}{c} \hat{z} \times \vec{r} \right)^2 - \frac{e^2}{r} + \frac{eB}{2mc} S_z + eEx . \]

(i) Describe the physics that this Hamiltonian describes (i.e., what is the system that this is applied? The system is subject to which fields, etc.).

(ii) Which terms should vanish if \( H \) is inversion (parity) symmetric?

(iii) Which terms should vanish if \( H \) has time reversal symmetry?

(iv) Which terms should vanish if \( H \) has rotational symmetry?

(e) Consider a particle in one-dimension. Show that all expectation values of the operator \( \hat{x} \hat{p} \) has an imaginary part. What is this imaginary part?

Hint: For a complex number \( z \), the imaginary part is given by \( \text{Im} z = (z - z^*)/2i \). Try to simplify \( \text{Im} (\hat{x} \hat{p}) \).

(f) For a free particle in one dimension, find the Heisenberg picture operators \( p_H(t) \) and \( x_H(t) \). Check the values of these expressions for \( t = 0 \).

Even though we usually treat photons as massless particles, it is possible that they have an extremely small, but non-zero mass, \( m_\gamma \), which we haven’t noticed up to now. Observational data of various kinds show no evidence of a non-zero value for \( m_\gamma \). On the other hand, we cannot prove that the mass is exactly zero by experimental means as the experiments always have errors and the physical effects of the photon mass can possibly be smaller than the errors. The best we can do is to place experimental upper bounds on the value of \( m_\gamma \). Thus, if \( m_\gamma \) is non-zero, it must be extremely small. In this problem, the photon mass effects on the hydrogen atom will be investigated, which will then be used to place an upper bound on \( m_\gamma \).

The existence of the photon mass changes the Coulomb interaction energy between the electron and the proton to the Yukawa form,

\[
V_{\text{Coulomb}} = -\frac{e^2}{r} \quad \rightarrow \quad V_{\text{Yukawa}} = -\frac{e^2}{r} e^{-r/\lambda},
\]

where \( \lambda = \hbar/m_\gamma c \) is the “reduced” Compton wavelength corresponding to the photon mass. The smallness of the photon mass translates into the largeness of the Compton wavelength as compared to the Bohr radius, i.e., \( \lambda \gg a_0 \).

(a) First of all, sketch the Coulomb and Yukawa potentials on the same graph and give one physical difference between them.

(b) Below, you will compute the effect of the Yukawa form on the Hydrogen atom by using first-order perturbation theory. For this purpose, expand the exponential term in the Yukawa potential (by assuming \( \lambda \) is large) and show that the Hamiltonian can be approximated as

\[
H \approx \frac{p^2}{2\mu} - \frac{e^2}{r} + c_1 + c_2 r = H_0 + H'
\]

where \( c_1 \) and \( c_2 \) are some constants and \( H' = c_1 + c_2 r \) can be treated as a perturbation. What are \( c_1 \) and \( c_2 \)?

(c) The bound-state wavefunctions of the hydrogen atom are given by \( \psi_{n\ell m}(r, \theta, \phi) = R_{n\ell}(r)Y_{\ell m}^m(\theta, \phi) \). For the special case of \( \ell = n - 1 \) (i.e., 1s, 2p, 3d, 4f, ... states), the radial wavefunction have the simple form

\[
R_{n,n-1} = N_n r^{n-\ell} e^{-r/n a_0} \quad (\ell = n - 1).
\]

Starting from the normalization relation for \( \psi_{n\ell m} \), derive the corresponding relation for the radial wavefunctions \( R_{n\ell} \). Show that the normalization constant for the \( \ell = n - 1 \) states are given as

\[
N_n = \left( \frac{2}{na_0} \right)^{n+\frac{3}{2}} \frac{1}{\sqrt{(2n)!}}.
\]

(d) Using perturbation theory, compute the corrections to the energies of the states with \( \ell = n - 1 \). (Note: The perturbed energies should be computed correctly in \( 1/\lambda \) and \( 1/\lambda^2 \) orders.)

(e) First consider the binding energy of the hydrogen atom. The best experimental value of the ground state (obtained from binding energy measurements) is

\[
E_{1s} = -13.598 434 48 \pm 9 \times 10^{-8} \text{ eV}.
\]

This value is consistent with the theoretically obtained value without taking the photon-mass effects into account. For this reason, if the photons really have mass, then the perturbation caused by the photon-mass effects must be smaller than the experimental error given above.

Using this observation, find a bound on the \( \lambda/a_0 \) ratio. Re-express this as a bound on the photon rest-energy, \( m_\gamma c^2 \) in eV units. (Note: \( \hbar c/a_0 \approx 3700 \text{ eV} \).)
(f) Photon-mass effects also cause small changes to the transition frequencies. Using the results in part (d), find the lowest order change in the $2p \to 1s$ transition frequency.

(g) The frequencies of $2p \to 1s$ transition lines have been measured with a fractional error of $10^{-11}$. The measurements are consistent with the theoretical calculations that exclude the photon-mass effects. Again, if the photon mass exists, then its effect on the transition frequency should be smaller than the experimental error.

Use this fact to find a bound on $\lambda/a_0$. Translate this to a bound on the rest energy of the photon.

You may need the following information:

- Gamma function integral: \( \int_0^\infty u^n e^{-su} du = \frac{n!}{s^{n+1}} \).

- Energy levels of the unperturbed H atom: \( E_n = -\frac{e^2}{2a_0 \ n^2} \).

[QM-2016-Nov] Q1: Answer the Following Questions.

(a) Consider a Dirac particle with mass \( m \).

(i) Compute the anti-commutator \( \{ \alpha_i, H_D \} = \alpha_i H_D + H_D \alpha_i \) as an operator expression.

(ii) Consider an energy eigenstate with energy \( E \). Starting with the expression \( \langle \{ \alpha_i, H_D \} \rangle \) show that

\[ \langle p_i \rangle = \frac{E}{c} \langle \alpha_i \rangle. \]

Interpret this result in terms of the momentum-velocity relation of relativistic particles.

(b) A particle in 1D that moves under the effect of a uniform force \( F \) has the Hamiltonian,

\[ H = \frac{p^2}{2m} - Fx. \]

Write down the Schrödinger equation for the momentum-space wavefunction, \( \phi(p') \) for energy \( E \), and solve it. After that, express the position-space wavefunction \( \psi(x') \) as an integral expression. 

Note: Do not try to evaluate the integral (the integral will be an Airy function).

(c) Consider the \( n \)th energy level of a particle in a 1D box of length \( L \). The force applied by the particle on one of the walls (say the wall on the right) can be computed in two different ways.

(i) We can consider the wall as an object with position coordinate \( L \) and “potential energy” given by \( E_n \), the total energy of the system. In that case, the force on the wall is given by

\[ F_n = -\frac{\partial E_n}{\partial L}. \]

Find the force \( F_n \).

(ii) Alternatively, we can treat the particle semiclassically, i.e., we consider it as a classical particle with energy \( E_n \), going back and forth in the box, periodically bumping and reflecting from the walls. Using this approach, compute the speed \( v_n \) of the particle, the impulse delivered on the right wall for each collision and the frequency of collisions with the right wall. Combining these compute the average force applied on the right wall.

Do not forget to compare your result with what you have found in part (i).

(d) Consider the non-relativistic Hydrogen atom with spin-orbit interaction. Which observables are conserved? Which quantum numbers (associated to which observables) can be used to label the energy eigenstates?
(e) Spin degrees of two spin-1 atoms are interacting with the Hamiltonian

$$H = -A \vec{S}_1 \cdot \vec{S}_2,$$

where $A > 0$ is a constant and $\vec{S}_i$ is the spin of the $i$th atom. Find the energy eigenvalues and their degeneracies?

(f) An electron in a Hydrogen atom is in the state

$$|\Psi\rangle = \frac{1}{\sqrt{3}} (\psi_{100} + \psi_{210} + \psi_{211}) |\uparrow\rangle,$$

where $\psi_{n\ell m}(r, \theta, \phi) = R_{n\ell}(r) Y^m_{\ell \phi}(\theta, \phi)$. Find

(i) $\langle L^2 \rangle$,

(ii) $\langle L_x \rangle$.

(g) Compute the ladder operator in the Heisenberg picture, $a_H(t)$ (Here $a$ is the ladder operator for 1D harmonic oscillator).

[QM-2016-Nov] Q2: Dipole Transitions for an anharmonic oscillator

The dipole selection rule states that a transition between two energy levels, $|\psi_n\rangle$ and $|\psi_m\rangle$, is possible if the position matrix element between these levels, $x_{nm} = \langle \psi_n | x | \psi_m \rangle$, is non-zero. The transition takes place with the emission or absorption of a single photon and the same condition holds for the forward, $\psi_n \rightarrow \psi_m$, as well as the backward transitions, $\psi_m \rightarrow \psi_n$. (In 3D, there is a transition if at least one of $x_{nm}$, $y_{nm}$ and $z_{nm}$ is nonzero.)

(a) Consider a 1D motion with the Hamiltonian

$$H = \frac{p^2}{2M} + V(x).$$

Let $|\psi_n\rangle$ be the energy eigenstates with eigenvalues $E_n$. Compute the commutator $[H, x]$. After that, by simplifying the expression $\langle \psi_n | [H, x] | \psi_m \rangle$, deduce that the momentum matrix elements are given by

$$p_{nm} = \frac{i M (E_n - E_m)}{\hbar} x_{nm}.$$

Note: By using this relation, we conclude that the dipole selection rule can also be expressed in terms of the momentum matrix elements.

(b) If the potential energy $V(x)$ has inversion symmetry, $V(x) = V(-x)$, then $|\psi_n\rangle$ are parity eigenstates. Which transitions are forbidden by the inversion symmetry? Why?

(c) Consider a harmonic oscillator in 1D,

$$H_0 = \frac{p^2}{2M} + \frac{1}{2} M \omega^2 x^2.$$

Using the dipole selection rule, determine the allowed transitions. In other words, if the system is at the $n$th energy level, to which levels it can make a single photon transition?
We usually model molecular oscillations by a harmonic oscillator, but some anharmonic effects are always present. Depending on presence of the inversion symmetry ($x \leftrightarrow -x$), we can model the anharmonic effects in two different ways:

(S) Some oscillators have inversion symmetry, for example the bending mode of the CO$_2$ molecule. In this case, we have $V(x) = V(-x)$ and the simplest Hamiltonian that includes anharmonic effects can be written as

$$H_S = \frac{\hat{p}^2}{2M} + \frac{1}{2} M \omega^2 x^2 + \lambda x^4,$$

where $\lambda$ is a small perturbation parameter.

(A) Some oscillators do not have inversion symmetry, for which the bending mode of the H$_2$O molecule is a good example. In this case, $V(x)$ can have a third order term and the Hamiltonian appears as

$$H_A = \frac{\hat{p}^2}{2M} + \frac{1}{2} M \omega^2 x^2 + \alpha x^3,$$

where $\alpha$ is also small.

Because of anharmonic terms, some of the forbidden transitions of the ideal harmonic oscillator may become allowed. In the remainder of this problem, you are going to identify these transitions.

(d) Consider the inversion symmetric case and the corresponding Hamiltonian $H_S$.

(i) By using perturbation theory, explicitly compute $x_{n,n+5} = \langle \psi_n | x | \psi_{n+5} \rangle$ up to (and including) first order in $\lambda$. Does this matrix element vanish?

Note: Take $x = x_0(a + a^\dagger)$ and do not express the constant $x_0$ in terms of the given parameters.

Your main purpose in here is to identify vanishing and non-vanishing matrix elements.

(ii) Identify all transitions $n \rightarrow m$ that are made possible by the anharmonic terms to order $\lambda^1$.

Note: The explicit computation of $x_{nm}$ is complicated and not needed for the rest of the problem. You only need to identify non-zero matrix elements in here; do not try to compute them all.

(iii) Suppose that you are looking at the emission spectrum of a hot gas with molecules that possess such an anharmonic mode. Sketch a plot of the intensity of emitted light vs frequency graph of such a gas.

(e) Now, consider $H_A$, i.e., the oscillators that lack inversion symmetry.

(i) By using perturbation theory, explicitly compute $x_{n,n+4} = \langle \psi_n | x | \psi_{n+4} \rangle$ up to (and including) first order in $\alpha$.

(ii) Identify all transitions $n \rightarrow m$ that are made possible by the anharmonic terms to order $\alpha^1$.

(iii) Sketch a plot of the intensity of emitted light vs frequency graph of a gas having molecules with such oscillators.

(f) Water molecule H$_2$O has three oscillation modes with frequencies given below.

<table>
<thead>
<tr>
<th>Modes</th>
<th>Frequency (THz)</th>
<th>Wavelength ($\mu$m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>mode-1</td>
<td>109.64</td>
<td>2.7344</td>
</tr>
<tr>
<td>mode-2</td>
<td>47.787</td>
<td>6.2708</td>
</tr>
<tr>
<td>mode-3</td>
<td>112.60</td>
<td>2.6625</td>
</tr>
</tbody>
</table>

Here, mode-3 has inversion symmetry while the other two modes lack that symmetry. Since these frequencies are in the infrared region, we do not normally perceive a color for water; it appears transparent. However, due to transitions made possible by the anharmonic effects, water molecules might possibly have some weak absorption in the visible range. As it is very weak, the absorption is noticeable only for large bodies of water, like the seas. There is a claim that the blue color of the seas is due to these weak absorption lines. Discuss if this explanation is plausible.

Visible range: 400 – 750 THz (0.4 – 0.7 $\mu$m)
[QM-2016-May] Q1: Answer the Following Questions.

(a) Let \( \vec{J} = \vec{L} + \vec{S} \) be the total angular momentum of a single electron. The state \( |\psi\rangle \) satisfies the following relations,
\[
J^2|\psi\rangle = \frac{3}{4} \hbar^2 |\psi\rangle, \quad J_z|\psi\rangle = \frac{1}{2} \hbar |\psi\rangle, \quad L^2|\psi\rangle = 2\hbar^2 |\psi\rangle.
\]
(i) Based on these, what can you say about the results of the following expressions?
\[
\begin{align*}
J^+|\psi\rangle &=? \\
S^2|\psi\rangle &=? \\
\vec{L} \cdot \vec{S}|\psi\rangle &=? \\
(J_z^2 + J_y^2)|\psi\rangle &=? \\
(L_z + S_z)|\psi\rangle &=?
\end{align*}
\]
(ii) If \( L_z \) is measured in this state, which values can be obtained? (Just list the outcomes, do not compute their probabilities.)

(b) Consider a particle with mass \( m \) in an infinite well of length \( L \). Suppose that there is a Dirac-delta bump in the middle of the box with the corresponding potential energy given by
\[
V_{\text{bump}}(x) = \lambda \delta(x - L/2),
\]
and the walls of the well are at \( x = 0 \) and \( x = L \). Treating \( V_{\text{bump}} \) as a perturbation, compute the first-order energy corrections for the ground state \( (n = 1) \) and the first excited state \( (n = 2) \).

(c) Consider a harmonic oscillator which is initially in state
\[
|\psi(t = 0)\rangle = \frac{1}{2}(|0\rangle + |1\rangle + \sqrt{2}|2\rangle).
\]
Find the average position \( \langle x \rangle_t \) at time \( t \).

(d) Consider the Hamiltonian
\[
H = \frac{1}{2m} \left( p_x - \frac{eB}{c} y \right)^2 + \frac{p_y^2}{2m}.
\]
(i) Which physical system does this Hamiltonian describe?
(ii) Discuss the invariance of \( H \) under the following symmetry transformations:
\begin{itemize}
  \item [\( \alpha \)] translation along \( x \),
  \item [\( \beta \)] translation along \( y \),
  \item [\( \gamma \)] time translation, and
  \item [\( \delta \)] rotation about \( z \) axis.
\end{itemize}
(iii) Using part (ii), decide on the conservation of the following quantities,
\[
p_x, \quad p_y, \quad H, \quad L_z = xp_y - yp_x.
\]

(e) Let \( \psi \) be the Dirac spinor of a particle.
(i) Show that \( J^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi \) satisfies the continuity equation.
(ii) Using \( J^\mu \) given above, express the probability density and discuss whether the probability density is positive definite or not.

(f) Let \( \phi(p') \) be the momentum-space wavefunction of a particle in 1D. How would you compute the expectation values \( \langle x \rangle \) and \( \langle p \rangle \) by using \( \phi(p') \)? Write down the expressions.

[QM-2016-May] Q2: Neutrino Oscillations
The 2015 Nobel prize in physics is awarded to Kajita and McDonald for their leading roles in experimental groups which demonstrated the neutrino oscillations. The neutrinos are created in one of three basic
flavors called electron-neutrino, $\nu_e$, muon-neutrino, $\nu_\mu$, and tau-neutrino, $\nu_\tau$. However, mass eigenstates (and hence energy eigenstates) of neutrinos are not identical with these flavor states. As a result, a neutrino created in one of these flavors will change its state with time and evolve into a superposition state having different flavors. Experiments have measured these changes in flavor. These results also imply that the neutrinos have mass.

The phenomenon can be understood as a simple application of time-dependence in quantum mechanics. To simplify the physics, we consider only two flavors, $\nu_e$ and $\nu_\mu$, and leave the spatial wavefunction out of the picture. This maps the problem to a two-level system problem like electron spin. Let $|\nu_1\rangle$ and $|\nu_2\rangle$ be the two energy eigenstates. The flavor states can be expressed as superpositions of these, and therefore in $2\times1$ matrix form.

$$|\psi\rangle = a|\nu_1\rangle + b|\nu_2\rangle \quad \rightarrow \quad \psi = \begin{bmatrix} a \\ b \end{bmatrix}.$$  

Let $\Delta$ represent the energy difference between these mass eigenstates. This enables us to take the Hamiltonian as

$$H = \frac{\Delta}{2} (|\nu_1\rangle\langle\nu_1| - |\nu_2\rangle\langle\nu_2|) \quad \rightarrow \quad H = \frac{\Delta}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  

As mentioned above, the two flavors are superpositions of the mass eigenstates. Let

$$|\nu_e\rangle = \cos \theta |\nu_1\rangle - \sin \theta |\nu_2\rangle,$$
$$|\nu_\mu\rangle = \sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle,$$

where $\theta$ is some angle (a constant of nature).

Muon-neutrinos are constantly created in the upper atmosphere. The Super-Kamiokande detector in Japan observes the flavors of these neutrinos.

(a) Suppose that a muon-neutrino is created in atmosphere so that the initial flavor state of the neutrino is

$$|\psi(t = 0)\rangle = |\nu_\mu\rangle.$$  

Let $t$ be the time the neutrinos travel until it reaches the detector. Find the state $|\psi(t)\rangle$ of the neutrino.

(b) The detector essentially measures the flavor of the neutrinos. In the language of quantum mechanics, we say that the detector measures the observable $D = |\nu_e\rangle\langle\nu_e|$. If the result is 1, the neutrino is $\nu_e$; and if the result is 0, it is $\nu_\mu$.

(i) Find the probability $P_t(\nu_\mu \rightarrow \nu_e)$ of detecting the neutrino as an electron-neutrino.

(ii) Sketch a plot of $P_t(\nu_\mu \rightarrow \nu_e)$ vs $t$.

(iii) What are the maximum and minimum values of $P_t(\nu_\mu \rightarrow \nu_e)$?

(iv) Does $P_t(\nu_\mu \rightarrow \nu_e)$ depend on time periodically? If so what is the period? (This is the reason for calling this phenomenon as an oscillation, of course.)

(c) Do the same of above for the probability $P_t(\nu_\mu \rightarrow \nu_\mu)$ of detecting the neutrino as a muon-neutrino.

(d) Let $m_1$ and $m_2$ be the masses of the neutrinos. If we assume that the created neutrino has a definite momentum $\vec{p}$, the energies of mass eigenstates are

$$E_i = \sqrt{m_i^2c^4 + p^2c^2}.$$  

These neutrinos are highly relativistic so that $E_1 \approx E_2 \approx pc \gg m_ic^2$. The small energy difference between energies $\Delta = E_1 - E_2$ arises from the mass difference. Find $\Delta$ as an approximate expression between masses $m_i$ and the neutrino energy $E \approx E_1 \approx E_2$. 
(e) The Super-Kamiokande detector can measure the energy \( E \) and the incidence direction and hence the length \( L \approx ct \) that the neutrinos travel inside the Earth. The flavor measurements are plotted as a function of \( L/E \) which clearly shows the oscillations. Using the period you have found in part (b-iv), and the data displayed in the graph above, compute a rough value for the mass-square difference, \( m_1^2 - m_2^2 \). (Use \( \text{eV}/c^2 \) unit for mass. Take \( h = 6.6 \times 10^{-16} \text{ eV s.} \))

Note: Because of this, experiments cannot directly measure the masses of neutrinos. Only differences of mass-squares can be obtained.

[QM-2015-Nov] Q1:
Answer the Following Questions.

(a) Carefully sketch the plots of the radial wavefunctions for the hydrogen atom states 1s, 2s and 2p.

(b) Let \( \vec{S} \) be the total spin of the particles that form a deuteron atom (i.e., one electron, one proton and one neutron, \( \vec{S} = \vec{S}_e + \vec{S}_p + \vec{S}_n \)). Which values can one possibly get if \( S^2 \) is measured?

(c) Consider the hydrogen atom, with spin-orbit coupling considered as a perturbation, \( H = H_0 + H_{\text{s.o.}} \), where

\[
H_0 = \frac{p^2}{2m} - \frac{e^2}{r} \quad \text{and} \quad H_{\text{s.o.}} = \frac{A}{r^3} \vec{L} \cdot \vec{S},
\]

where \( A \) is some positive constant. Consider the \( n = 2 \) levels of the unperturbed atom, namely the 2s and 2p states, which are 8-fold degenerate. Spin-orbit coupling will split this degeneracy. Briefly discussing first-order perturbed energies, qualitatively describe how the levels will split (in other words, what are the quantum numbers and degeneracies of each level?)

Note: Do not evaluate the radial integrals.

(d) Consider the following Hamiltonian

\[
H = \frac{p^2}{2m} - \frac{e^2}{r} + \frac{A}{r^3} \vec{L} \cdot \vec{S} + \mu_B B(L_z + 2S_z) + eEz,
\]

where \( A, B \) and \( E \) are constants.

(i) Which of these constants should be zero if \( H \) has inversion (parity) symmetry?

(ii) Which of these constants should be zero if \( H \) has time-reversal symmetry?

(iii) Does \( H \) have any rotational-symmetry if all three of these constants are non-zero?

(e) A “velocity operator” \( \vec{v}_{op} \) can be defined in quantum mechanics by using the relationship

\[
\langle \vec{v}_{op} \rangle = \frac{d}{dt} \langle \vec{r} \rangle.
\]
(i) Derive the velocity operator for a relativistic Dirac particle.
(ii) What are the eigenvalues of, say, the \(x\)-component of the velocity, \(v_{x, \text{op}}\)?
(iii) Is there a maximum eigenvalue for a component of the canonical momentum, say \(p_x\)?
(iv) Discuss why it is reasonable that \(v_{x, \text{op}} \neq \frac{p_x}{m}\) for a relativistic particle.

(f) Consider a free particle in 1D with the Hamiltonian \(H = \frac{p^2}{2m}\). Compute the position operator in the Heisenberg picture, \(x_H(t)\).

Consider a harmonic oscillator
\[
H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2.
\]
Let \(\beta\) be a complex number. The state \(|\beta\rangle\) defined as
\[
|\beta\rangle = Ne^{\beta a^\dagger}|0\rangle = N \left(|0\rangle + \beta|1\rangle + \frac{\beta^2}{\sqrt{2}}|2\rangle + \cdots + \frac{\beta^n}{\sqrt{n!}}|n\rangle + \cdots \right) = N \sum_n \frac{\beta^n}{\sqrt{n!}}|n\rangle
\]
is called a coherent state. Coherent states have some interesting physical properties which makes them useful in the description of various physical phenomena. In this problem, we will see some of these properties.

(a) Find the normalization constant \(N\).

(b) Show that \(a|\beta\rangle = \beta|\beta\rangle\). Use this, and the associated bra equation \(\langle\beta|a^\dagger = \beta^*\langle\beta|\) to compute the following expectation values in terms of \(\beta\).
\[
\langle a \rangle , \quad \langle a^\dagger \rangle , \quad \langle a^\dagger a \rangle , \quad \langle aa^\dagger \rangle , \quad \langle a^2 \rangle , \quad \langle a^{12} \rangle , \quad \langle a + a^\dagger \rangle , \quad \langle (a + a^\dagger)^2 \rangle , \quad \langle \frac{a - a^\dagger}{i} \rangle , \quad \langle \left(\frac{a - a^\dagger}{i} \right)^2 \rangle,
\]
\[\text{Note-1: Let } \beta = \beta_R + i\beta_I \text{ where } \beta_R = \text{Re } \beta \text{ is the real part and } \beta_I = \text{Im } \beta \text{ is the imaginary part of } \beta. \text{ You may want to express some of the above in terms of } \beta_R \text{ and } \beta_I.\]
\[\text{Note-2: 1 point for each expression. This means that all of these are straightforward. Means: if you are spending too much time on any one of them, you are doing something wrong.}\]

(c) Write down \(\langle x \rangle, \langle p \rangle, \Delta x \text{ and } \Delta p\). Finally, check the uncertainty principle. Is this a “minimum uncertainty state”?

(d) Suppose that the oscillator is in a coherent state \(|\beta_0\rangle\) at time \(t = 0\) where the parameter \(\beta_0\) is a real number,
\[
|\psi(t = 0)\rangle = |\beta_0\rangle .
\]
Show that, at time \(t\), the state is still a coherent state,
\[
|\psi(t)\rangle = e^{-i\theta(t)}|\beta(t)\rangle
\]
where \(\theta(t)\) is some overall phase angle and \(\beta(t)\) is a time-dependent complex parameter. Find both \(\theta(t)\) and \(\beta(t)\).

(e) For the state given in part (d), write down \(\langle x \rangle_t, \langle p \rangle_t, \Delta x_t, \Delta p_t\) for the state at time \(t\). Do the uncertainties depend on time? How do you physically interpret the time dependence of \(\langle x \rangle_t\) and \(\langle p \rangle_t\)?

(f) Write down \(\langle H \rangle_t\). Does it depend on time? What is the meaning of this?

[QM-2015-May] Q1:
Answer the Following Questions.
(a) For a particle in an infinite 1D well, carefully sketch the plots of the energy eigenfunctions \( \phi_n(x) \) and the associated probability densities for position measurements for the lowest three eigenstates.

(b) Consider a harmonic oscillator with frequency \( \omega \). Suppose that the frequency of the oscillator has been increased by \( \omega \rightarrow \omega \sqrt{1 + \epsilon} \) where \( \epsilon \) is a small number.

(i) Think of this as a perturbation problem: Write down the unperturbed Hamiltonian \( H_0 \) and the perturbation \( H' \). Then, use the perturbation theory to compute the change in energy eigenvalues to first order in \( \epsilon \). (Consider a general level \( n \).)

(ii) Find the exact energy eigenvalues and compare with the results in subpart (i).

(c) Describe and compare the three pictures, i.e., Schrödinger, Heisenberg and Interaction pictures.

(d) Suppose that \( |\alpha\rangle \) and \( |\beta\rangle \) are parity eigenstates. The position matrix element \( \langle \alpha|x|\beta\rangle \) vanishes for some values of the parity eigenvalues. State for which eigenvalues this happens and then give a short proof.

(e) Consider a particle in 3D in the state

\[ \psi = f(\theta, \phi) e^{ikr}/r \]

where \( r, \theta, \phi \) are the spherical coordinates, \( k \) is a positive constant and \( f \) is an arbitrary function. First computing the probability-current density \( \vec{J} \), evaluate the total probability current leaving in the radial direction and show that

\[ \oint \vec{J} \cdot d\vec{S}_{\text{out}} = \frac{\hbar k}{m} \oint |f(\theta, \phi)|^2 d\Omega. \]

(f) The 1s wavefunction for the hydrogen atom is

\[ \psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi a_0^3/2}} e^{-r/a_0}. \]

Compute \( \langle 1/r \rangle \). Hint: \( \int_0^\infty x^n e^{-x} dx = n! \)

(g) Explain briefly why the Dirac Hamiltonian has to be linear in momentum.

**[QM-2015-May] Q2: Spin-Orbit Coupling**

**Note:** Some parts of this question can be solved independently from the rest of the problem.

(a) Consider an electron (charge \(-e\)) moving inside an electric field \( \vec{E} \). Using the Lorentz transformation of electromagnetic fields, show that the electron also experiences the so-called spin-orbit interaction

\[ H_{S.O.} = -\frac{e}{m^2 c^2} \vec{E} \cdot (\vec{S} \times \vec{p}). \]

Give clear explanation of the physics behind.

**Notes:** (1) Disregard effects associated with the Thomas precession. (2) Magnetic moment of electron in Gaussian units is \( \vec{\mu} = -(e/mc)\vec{S} \).

(b) Suppose that a non-relativistic electron is moving under the effect of a potential energy

\[ U(\vec{r}) = U(x, y, z) = \frac{1}{2}m\omega^2 z^2, \]

i.e., the electron is free on \( xy \) plane and there is harmonic binding along \( z \). Work out the Hamiltonian and show that

\[ H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{1}{2}m\omega^2 z^2 + \alpha z(p_x\sigma_y - p_y\sigma_x). \]

What is the value of \( \alpha \)? What is the dimension of \( \alpha \) (i.e., its unit)?

**Hint:** \( U \) can be thought as electrostatic in origin.
(c) Let \( \vec{L} = \vec{r} \times \vec{p} \) be the orbital angular momentum and \( \vec{J} = \vec{L} + \vec{S} \). Consider all components of the following vector quantities: \( \vec{p}, \vec{r}, \vec{L}, \vec{S}, \vec{J} \). Which components of these vectors are conserved? Do not try to compute \( 5 \times 3 = 15 \) different commutators! Give the answer by using symmetry arguments or by investigating the dependence of \( H \) on particular operators.

(d) Let us look for common eigenstates of \( H, p_x \) and \( p_y \).

\[
H \psi = E \psi ,
\]
\[
p_x \psi = \hbar k_x \psi ,
\]
\[
p_y \psi = \hbar k_y \psi .
\]

Working out the energy eigenvalue equation, show that it reduces to a spin-dependent 1D problem along the \( z \)-direction

\[
\left( \frac{p_z^2}{2m} + \frac{1}{2} m \omega^2 z^2 + A z \vec{\sigma} \cdot \hat{n} \right) \psi = E' \psi ,
\]

where \( A \) is some constant and \( \hat{n} \) is some constant unit vector. Find \( A \) and \( \hat{n} \) in terms of \( k_x, k_y \) and \( \alpha \). Define \( E' \).

(e) It therefore appears that \( \psi \) is separable, so it can be written in the form

\[
\psi = \phi_1(x) \phi_2(y) \phi_3(z) \phi_{\text{spin}}
\]

where \( \phi_{\text{spin}} \) is the spin state (i.e, \( 2 \times 1 \) column matrix).

(i) Write down the eigenvalue equations satisfied by each of the four individual parts, \( \phi_1, \cdots, \phi_{\text{spin}} \), and identify the quantum numbers associated with each equation.

(ii) Provide a short, but complete description of the physical state of each function \( \phi_1, \cdots, \phi_{\text{spin}} \).

(iii) Write down the energy eigenvalue \( E \) in terms of the quantum numbers involved. 

Hint: The matrix \( \vec{\sigma} \cdot \hat{n} \) has eigenvalues \( \pm 1 \).
[AN-2017-May] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

(a) Determine the types of constraints (time-dependent/time-independent holonomic, nonholonomic etc.) in the systems defined below. When suitable, (a1) write down the constraint functions and (a2) give a set of generalized coordinates to define the motion of the systems in each case.

(i) A cylinder rolling without sliding on an inclined plane.

(ii) A stick of length $\ell$ and mass $m$ moves on a smooth horizontal plane with a constant acceleration $\mathbf{a}$ while it is being in uniform rotation (with angular frequency $\Omega$) about a vertical axis passing through the center of mass of the stick.

(iii) Two point masses on a plane whose relative distance is known as $f(t)$.

(iv) A vertical disk rolling on a horizontal plane.

(b) A particle of mass $m$ moves on the curve $z = h(x)$ under a gravitational field, $\mathbf{g} = -g\hat{z}$. Obtain the force of constraint(s) in magnitude by using the Lagrangian formulation only.

(c) (i) Suppose that there is a net force on a system of particles (with a total mass $M$) is $\mathbf{F}$ and in a fixed frame $K'$ the position and velocity of the center of mass (CM) are $\mathbf{R}$ and $\mathbf{V}$, respectively. If $L_S(N_S)$ represents angular momentum (torque) in a frame $S = K'$ or CM, show that the following relations are hold:

(a) $L_{K'} = L_{CM} + MR \times V$,
(b) $N_{K'} = N_{CM} + R \times F$,
(c) $\dot{L}_{CM} = N_{CM}$.

(ii) Consider a system of particles with total mass $M$. The position vector of the center of mass of the system is $\mathbf{R}$ in a stationary frame $K'$. There is a moving point $K$ whose position vector is $\mathbf{r}_K$. If $\mathbf{L}_K$ is the angular momentum of the system about the point $K$, show that

$$\dot{\mathbf{L}}_K = \mathbf{N}_K - M(\mathbf{R} - \mathbf{r}_K) \times \ddot{\mathbf{r}}_K$$

where $\mathbf{N}_K$ is the torque on the system with respect to the point $K$. Under what conditions the above formula reduces to the conventional one? Interpret this physically.

(d) Two beads of masses $m_1$ and $m_2$, which are free to slide along a massless wire on a smooth horizontal table, are connected by a massless spring with constant $k$ as shown in the figure. The wire rotates with a constant angular velocity $\Omega$ about an axis shown.

(i) By using the wire’s frame (rotating frame) and the modified Newton’s 2nd law $(m \mathbf{a}_r = \mathbf{F}_{eff})$, obtain the equations of motion for each bead.

(ii) Obtain the reaction forces on each bead.

(iii) Using the coordinates of the center of mass and the relative position, decouple the differential equations in part i. Find the positions of the beads as a function of time.

(iv) Express the total reaction force in terms of the coordinates in part iii.

(e) A bead of mass $m$ is free to move on a massless smooth horizontal wire. The wire is in uniform rotation with constant angular velocity $\Omega$ about a vertical axis passing through one end of the wire as shown in the figure.

(i) Write down the Lagrangian of the system.

(ii) Construct the Hamiltonian and obtain the Hamilton-Jacobi (HJ) equation.

(iii) Using only the HJ equation found in part ii, find the position of the bead as a function of time.
(f) (i) Show that the volume elements in phase space are invariant under canonical transformations, that is, \( dq_j dq_j = dQ_k dP_k \) if \( (q_j, p_j) \) pairs are related to \( (Q_k, P_k) \) pairs by a canonical transformation.

(ii) Consider the following transformation

\[
Q_1 = q_1, \quad Q_2 = p_2, \\
P_1 = p_1 + \alpha p_2, \quad P_2 = \beta q_1 - q_2
\]

where \( \alpha \) and \( \beta \) are some constants. To make the transformation canonical, determine the condition(s) on \( \alpha \) and \( \beta \).

(iii) Find a suitable generating function.

Hint: Consider using the exactness method with \( (p_1, q_2, Q_1, Q_2) \) as independent variables to obtain a mixed generating function.

[AN-2017-May] Q2: Apside Down

A theorem in central force systems states that “the only force laws yielding closed orbits for all bounded motions are the linear and inverse-square forces”. To show this, there are three main steps to follow: (1) finding potentials having constant apsidal angles \( (\theta_A) \), the angle between the minimum and maximum position vectors from the force center (as shown in the figure), (2) expressing the apsidal angle for near-circular orbits, and finally (3) finding the potentials giving constant apsidal angles as rational multiple of \( \pi \) for general non-circular orbits. Let us go through the steps to achieve the goal. Consider the motion of a particle with reduced mass \( \mu \) under the influence of a central force with potential energy \( U(r) \).

(a) Write down the Lagrangian and Lagrange’s equations of the particle.

(b) Find two constants of motion and explain in physical terms why they are expected to be conserved.

(c) Show that the motion can be described effectively in one-dimension under the influence of an effective potential energy \( V_{\text{eff}}(r) \). Find \( V_{\text{eff}}(r) \).

(d) Show that the radial equation can be put into the form

\[
\frac{1}{2\tilde{\mu}} \left( \frac{du}{d\theta} \right)^2 + V_{\text{eff}}(u) = E,
\]

where \( E \) is the total energy and \( u \equiv 1/r \). Find \( \tilde{\mu} \).

(e) Derive the condition for a circular orbit at \( u = u_0 = \text{const.} \) by using \( V_{\text{eff}}(u) \) found in part (d).

(f) Now assume a small perturbation around the equilibrium \( u = u_0 \) such that \( u = u_0 + \epsilon \eta(\theta) \) (\( \epsilon \) is the control parameter). Express the energy \( E \) in part (d) by expanding \( V_{\text{eff}}(u) \) around \( u_0 \). Keep the terms up to and including \( O(\epsilon^2) \).

(g) It is known that the solution of a one-dimensional simple harmonic oscillator (SHO) is \( x(t) = A \cos(\omega t) \) with the total energy \( E_{\text{SHO}} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} kx^2 \) with \( \omega^2 = k/m \). Comparing the energy \( E \) in part (f) with \( E_{\text{SHO}} \), write down the solution \( u(\theta) \). Find \( \Omega \) in terms of \( U(u) \) and its derivatives.

(h) Using the definition of the apsidal angle \( (\theta_A) \), show that \( \theta_A = \pi/\Omega \).
The arguments so far are valid for only small deviations from the circular orbit. To get a constant $\theta_A$ for any value of $E$, $\Omega$ should be a positive constant for any distance $r$. Then determine the form of $U(r) = cr^d$, namely express $c$ and $d$ in terms of the parameters. Find the apsidal angle $\theta_A$.

For $d > 0$, consider the effective potential $V_{\text{eff}}(u)$ and sketch it as a function of $u$. By using the limiting form of the energy expression in part (d) for very large $E$ values and with the help of the SHO analogy determine $\theta_A$. From the form of $\theta_A$ in part (i), find the value of $d$ and the potential $U(r)$. What type of force does the potential energy $U(r)$ correspond to?

For $d < 0$, sketch the effective potential $V_{\text{eff}}(u)$. In this case consider $E \to 0$ limit in the energy expression. By making a substitution, put the equation into the SHO form and identify the angular frequency. Obtain $\theta$ and the value of $d$. What type of force does the potential energy $U(r)$ correspond to? Comment on $d = 0$ case.

[AN-2016-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

(a) A particle of mass $m$ is making a straight-line motion, say along the $y$ axis, with a displacement of $\Delta y = y_2 - y_1$ in a $\Delta t = t_2 - t_1$ time interval. The time averaged kinetic energy of the particle is

$$\langle T \rangle = \frac{1}{\Delta t} \int_{t_1}^{t_2} \frac{1}{2} m \dot{y}^2 \, dt,$$

where $\dot{y}$ is the time derivative of $y$. Determine the position of the particle, $y(t)$, as a function of time in terms of the given parameters so that the average kinetic energy $\langle T \rangle$ gets its minimum value.

(b) A particle of mass $\mu$ moves in a central force system with a potential energy $U(r) = \kappa r^4$.

(i) For the particle to have a circular orbit of radius $r_0$, what angular momentum $\ell$ and total energy $E$ should it have? Express them in terms of the given parameters.

(ii) Find the condition on $\kappa$ for the mass $\mu$ to have a stable circular orbit at $r_0$.

(iii) If a very small radial kick is given to the particle at $r_0$, determine whether the orbit of the subsequent motion is closed or not. Explain.

(c) There is a bead of mass $m$ along a massless rod which makes a constant angle $\theta_0$ with the horizontal plane. The bottom end of the rod is fixed at the point $O$ on the horizontal plane, and the rod is kept rotating uniformly with angular velocity $\omega$ about the vertical axis passing through $O$. The system is under a uniform gravitational field $\vec{g} = -g \hat{z}$ and there is no friction.

(i) By using the rotating frame and the modified Newton’s 2nd law $(m \ddot{a}_r = F_{\text{eff}})$, determine the value of $\omega$ to keep $m$ at rest at a distance $d$. Express also the normal force in terms of the given parameters.

(ii) Repeat the above part by applying the Newton’s 2nd law in the inertial (fixed) frame.

(d) A thin uniform disc of radius $R$ and total mass $M$ lies on a horizontal surface.

(i) Calculate the inertia tensor $J_{ij}$ of the disc in the given coordinate system.

(ii) If the disc is rotated about the $z$ axis with a uniform angular velocity $\omega$, calculate the kinetic energy of the disc.

*Hint: Think of using the parallel axis theorem.*

(e) Consider the following transformation from $(q, p) \to (Q, P)$,

$$Q = q^\alpha \cos(\beta p), \quad P = q^\alpha \sin(\beta p),$$

where $\alpha$ and $\beta$ are some constants.
(i) Determine the values of $\alpha$ and $\beta$ so that the transformation is canonical.
(ii) Find a generating function of type $F_3$.

8 (f) There is a particle of mass $m$ falling under a uniform gravitation field $g$.
(i) Write down the Hamilton-Jacobi equation for the system.
(ii) Determine the motion of the particle by solving the Hamilton-Jacobi equation in part (i).

[AN-2016-Nov] Q2:

A pointlike object of $m_1$ is constrained to slide along a vertical massless shaft. One end of a massless spring of constant $k$ is connected to $m_1$ while the other end is attached to the top of the shaft. A massless string of length $\ell$ connects $m_1$ to a pointlike object of mass $m_2$ as shown in the figure. The shaft is set in uniform rotation with angular speed $\Omega$. The system is in a uniform gravitational field $g$. There is no friction anywhere in the system. Ignore the relaxed length of the spring.

(a) Write down the Lagrangian of the system by choosing suitable generalized coordinate(s).
(b) Obtain the Euler-Lagrange equation(s) of motion.
(c) Using part (b), determine the equilibrium configuration(s) if there is(are) any. Interpret your result physically.

In the rest of the problem, assume that $k \to \infty$, that is, the spring is infinitely stiff.

(d) Reduce the equation(s) in part (b). Interpret your result physically.
(e) Using part (d), discuss the stability of the equilibrium(s) by making an expansion around it/them. Determine the condition(s) on $\Omega$ from the stability point of view. Find the frequency of small oscillations whenever relevant.
(f) Construct the Hamiltonian of the system and obtain the Hamilton’s equations of motion.
(g) Show that from the Hamilton’s equations of motion, the equation(s) found in part (d) follow(s).
(h) Discuss the following and in each case explain why:
   (i) Is the Hamiltonian a constant of motion?
   (ii) Is the Hamiltonian equal to the total energy of the system?
   (iii) Is the total energy of the system conserved? Compute $dE/dt$ in terms of the given parameters and generalized coordinates.

[AN-2016-May] Q1:
Answer the Following Questions
(a) It is known that equations of motion remain to be invariant under the transformation $L(q, \dot{q}, t) \rightarrow L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d\Lambda(q, t)}{dt}$. That is, both $L$ and $L'$ describe the same physics. One can define the Hamiltonian as $H = p\dot{q} - L$.

(i) Express the relation between the transformed conjugate momentum $p'$ and $p$.

(ii) Construct the relation between the transformed Hamiltonian $H'$ and $H$.

(iii) Show the invariance of the Hamilton's equations.

(b) Consider a simple pendulum with a point mass $m$ attached to one end of a string of length $l$. The pendulum swings in a vertical plane under uniform gravitational field. Obtain the equation of motion by only using the D'Alembert's principle.

(c) Consider the following transformation $Q = p^m q'^m$ and $P = p^n q'^n$ where $m, m', n, r$ are some integers.

(i) For the transformation $(q,p) \rightarrow (Q, P)$ to be canonical, determine the condition(s) among the parameters.

(ii) If $m \neq 0$, find a generating function.

(d) A rod of length $2\ell$ and mass $m$ is hinged from its center of mass to a vertical massless shaft. The lower end of rod is connected to the shaft with a spring of stiffness $k_1$. There is another spring with stiffness $k_2$ connecting the shaft and $\ell/2$ away from the hinge point. The system is in rotation with a constant angular speed $\omega$ about the axis along the shaft. The relaxed length of the springs are negligible. The sketch of the problem is given in the figure. Using the Euler's equation for the rigid body motion,

(i) obtain the equation of motion,

(ii) find the amount of displacement of the spring with stiffness $k_1$ when the system is in equilibrium.

(e) Consider the Lagrangian

$$L = \frac{1}{2}m\dot{q}_j^2 - \frac{1}{2}kq_j^2 - \beta(q_1 q_2 + q_2 q_3 + q_3 q_1), \quad j = 1, 2, 3 \quad \text{and} \quad m, k, \beta$$

are parameters. Consider an infinitesimal rotation about an axis passing through the origin and the point $(1,1,1)$ such that $q_j \rightarrow q_j + \epsilon(n \times q)_j + O(\epsilon^2)$ where $n$ is the vector along the axis of rotation. Keeping terms up to order of $\epsilon$.

(i) Show that the Lagrangian $L$ is invariant under the given rotation.

(ii) Check also that $\frac{\partial L}{\partial \dot{q}_j}(n \times q)_j$ is indeed constant. Note that there is a sum over repeated indices.

[AN-2016-May] Q2:

A massless vertical ring of radius $R$ is set in rotation with a constant angular speed $\omega$ about the vertical axis passing through its center. There are two identical pointlike beads of mass $m$ which are bound to the ring but otherwise free to slide on the ring smoothly. They are also connected by each other through a stiff massless spring with constant $k$ and with negligible relaxed length. The beads start their motion from the top of the ring and the spring is stiff enough to remain horizontal and in-plane with the ring at all times. The system is in a uniform gravitational field $g$. The motion of the system is depicted in figure 1. A frontal view of the motion is shown in figure 2.

(a) By choosing appropriate generalized coordinates, construct the Lagrangian of the system.

(b) Obtain Lagrange’s equation(s) of motion.

(c) Using part (b), determine the equilibrium point(s) of the system.

(d) By making an expansion around equilibrium point(s) found in part (c), discuss the stability of the motion for all possible values of $\omega$.

(e) If the system possesses stable equilibrium point(s), find the frequency of small oscillations.
(f) If the beads are at the top position at $t = 0$, express the angular speed of one of the particles along the ring at a later time in terms of the given parameters and generalized coordinate(s) only (no generalized velocities).

(g) From the Lagrangian in part (a), construct the Hamiltonian $H$ and obtain Hamilton’s equations. Show that they reduce to the one(s) in part (b).

(h) Is $H = \text{constant}$?, Is $H = E$? Explain each clearly and show your steps. Compute $dE/dt$ in terms of given parameters and generalized coordinates. Discuss whether it’s constant or not.

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[AN-2015-Nov] Q1:
Answer the Following Questions.

(a) How many independent degrees of freedom does a rigid body have for the following cases? Explain each carefully.
   (i) A translational motion in space.
   (ii) A planar motion in space.
   (iii) A rotation about an axis fixed in space.
   (iv) A general motion in space.

(b) Consider the generating function $F(q, P) = qP + P^3/(6m^2g)$ for a particle of mass $m$ falling under gravitational field $g$.
   (i) Find $p$ and $Q$ in terms of $q$ and $P$. Show that the transformation is canonical.
   (ii) Find the new Hamiltonian.
   (iii) Obtain Hamilton’s equations and solve them.

(c) Find the force field $F(r)$ for the following trajectories ($c, d, f, g, \epsilon$ are some constants): 
   (i) $r = \frac{1}{c + d\theta}$
   (ii) $r = \frac{f^2}{1 - \epsilon \sin 2\theta}$
   (iii) $r = \frac{g^2}{1 - \epsilon \cos \theta}$
(d) Show that for a conservative system of \( n \) particle the kinetic energy of the system can also be written as \( T = \frac{1}{2} \sum_j \dot{q}_j p_j \) where \( p_j \) is the generalized momenta.

*Hint: Think of the general transformation from Cartesian to generalized coordinates and express \( T \) in generalized coordinates and then reduce it when the system is conservative.*

(e) Consider a Galilean transformation, \( \vec{r}' = \vec{r} - \vec{v}_0 t \) between inertial frame \( K \) and \( K' \) for a particle of mass \( m \) under a potential energy \( U(\vec{r}_1 - \vec{r}_2) \) with \( |\vec{r}_1 - \vec{r}_2| \) as interparticle separation. If \( L(L') \) is the Lagrangian in the \( K(K') \) frame, express \( L' \) in terms of \( L \). Show that they describe the same physics, i.e., the equations of motion are invariant under Galilean transformation.

(f) Consider the differential equations, \( \dot{q} = a p + b q \), \( \dot{p} = c q - q^2 \). Find the condition(s) on the constants \( a, b, c \) to have \( q \) and \( p \) as canonical pair. Find the Hamiltonian \( H \) and obtain the corresponding Lagrangian.

(g) Consider two-body central force motion of a system with force field \( F(r) \). Show that the system can be treated as effectively one-dimensional problem with the equation of motion, \( \mu \ddot{r} = F_{\text{eff}}(r) \), where \( \mu \) is the reduced mass. Express \( F_{\text{eff}}(r) \) in terms of \( F(r) \) plus other term(s). Interpret \( F_{\text{eff}} \) physically.

[AN-2015-Nov] Q2: Oscillating Rod

A uniform rod of length \( 2b \) and mass \( m \) remains in equilibrium on the top of a rough semicircular fixed cylinder of radius \( R \) as shown in figure. Once given a small vertical tap to the rod, it can oscillate back and forth on the cylinder under uniform downward gravitational field.

(a) Calculate the moment of inertia of the rod about an axis passing through its center of mass.

(b) Write down the Lagrangian and obtain Lagrange’s equation(s) of motion.

(c) By assuming small amplitude of oscillations, simplify the Lagrange’s equation(s) in part (b).

(d) By using the result in part (c), obtain the frequency of small oscillations and the period of the motion.

(e) Now, obtain the equation(s) of motion by using the Euler’s equations and compare with the one(s) in part (b).

(f) From the Lagrangian in part (b), construct the Hamiltonian \( H \) and obtain Hamilton’s equations. Show that they reduce to the one(s) in part (b).
(g) Is $H = \text{constant}$?, Is $H = E$? Is $E = \text{constant}$? Explain each clearly.

(h) Get the equation(s) of motion by using Newtonian mechanics. Do you agree with what you get in part (b)?

[AN-2015-May] Q1:
Answer the Following Questions.

(a) Explain briefly whether the following systems are scleronomic (fixed time constraint) or rheonomic (explicit time-dependent constraint), holonomic or nonholonomic, and conservative or non-conservative. *Hint: Holonomic means constraints with functions of coordinates only, no velocities.*

(i) A cylinder rolling down on a fixed cylinder and no friction between them. Include the fact that the rolling cylinder will leave the fixed one.

(ii) A bead moving along a long wire which rotates with a constant angular velocity $\omega$ in a vertical plane about a horizontal axis. No friction in the system.

(iii) A point particle constrained on the rough inner surface of a hyperboloid.

(iv) A cylinder rolling down on a rough inclined plane.

(v) A particle confined to be inside of empty sphere.

(b) Prove that the following transformations are canonical

(i) $Q = pq^2$, $P = \frac{1}{q}$.

(ii) $Q = \log \left(1 + \sqrt{q} \cos(p)\right)$, $P = 2\sqrt{q} \left(1 + \sqrt{q} \cos(p)\right) \sin(p)$.

(iii) Find a generating function for the transformation in part (i).

(c) The time evolution of the angular momentum $\vec{L}$ can be given in the Hamiltonian formulation $\frac{d\vec{L}}{dt} = \{\vec{L}, H\}$. By using this equation, show explicitly that for a central force field system, the angular momentum $\vec{L}$ is conserved.

(d) Consider the following differential equations,

$$\dot{q} = ap + bq, \quad \dot{p} = cq + dp,$$

where $a, b, c, d$ are some constants. Find the conditions on these constants for these equations to be Hamilton’s equations and construct the Hamiltonian explicitly. Obtain the corresponding Lagrangian.

(e) Consider the Lagrangian of a free particle of mass $m$, $L = \frac{1}{2}m\dot{\vec{v}}^2$. Obtain the equation(s) of motion. Now find the Lagrangian under the following transformation $x' = x \cos \theta + y \sin \theta$, $y' = -x \sin \theta + y \cos \theta$, $z' = z$, where $\theta = \theta(t)$ is given. Find the Lagrangian in transformed coordinates. From the equations of motion, check whether there is any “force” on the particle. If so, explain what they are.

(f) For a rigid body the formula for the angular momentum vector $\vec{L}$ can be given as $\vec{L} = I\vec{\omega}$ where $I$ is the principal moments of inertia of the rigid body and $\vec{\omega}$ is the angular velocity. As we know that even though $\vec{L}$ and $I$ are both constant, $\vec{\omega}$ is not constant, i.e., $\dot{\vec{\omega}} \neq 0$. Explain clearly why it is still okay for $\vec{\omega}$ not to be a constant.

(g) Consider a simple Atwood’s machine with masses $m_1$ and $m_2 > m_1$ at the end of an inflexible string of length $\ell$. Using the Lagrangian formulation, find the tension in the string. Compare your result by following the Newtonian approach.

(h) First state the condition(s) for the Hamiltonian $H$ to be equal to the total energy $E$ of the system. Then, do all the steps to show at what stage we need these conditions to make sure that $H = E$.

[AN-2015-May] Q2: Two Coupled Oscillators
A rigid rod pendulum of mass $m_1$ and length $\ell$ whose free end is connected to a mass-spring harmonic oscillator (mass $m_2$ and spring constant $k$). The coupling between them is provided by another identical spring as shown in figure. The pendulum is assumed to swing with small amplitude under uniform gravitational field so that the springs and the point mass $m_2$ remain to move horizontally at all times. When the rod is vertical and $m_2$ is at $d/2$, both of the springs are relaxed. In the rest of the problem assume $m_1 = m_2 = m$ and also that $k = mg/(3\ell)$. Assume that the springs are massless.

(a) Using appropriate generalized coordinates, write down the kinetic and potential energies of the system.

(b) Using part (a), to linearize the equations of motion put the kinetic energy into the quadratic form, i.e., $T = \frac{1}{2}M_{ij}\dot{q}_i\dot{q}_j$. Construct the matrix $M$.

(c) Write down the potential energy of the system and put it into the quadratic form $U = \frac{1}{2}V_{ij}q_iq_j$. Construct the matrix $V$.

(d) Show that the secular (characteristic) equation can be written in the factorized form

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = 0,$$

where $\omega_1^2 = \frac{1}{2}g/\ell$ and $\omega_2^2 = \frac{8}{3}g/\ell$.

(e) Find the corresponding eigenvectors $\vec{a}_1$ and $\vec{a}_2$. Using them construct the transformation matrix $P$ such that $\vec{q} = P\vec{\eta}$. Here $\vec{q} = (q_1 \ q_2)^T$ are the generalized coordinates chosen initially and $\vec{\eta} = (\eta_1 \ \eta_2)^T$ are the normal coordinates. Normalize the eigenvectors using $P^TMP = 1$. Check the combination $P^TVP$.

(f) Using the given information in part (e), show that $\vec{\eta} = P^T\vec{q}$. Obtain the normal coordinates in terms of the originally chosen ones. Interpret each.

(g) Obtain $\eta_1(t)$ and $\eta_2(t)$ and express original coordinates as a function of time for arbitrary initial conditions.

(h) Find the positions of the rod and the point mass as a function of time if $m_2$ is at the equilibrium point being at rest initially while the tip of the rod is at $d/8$ with no initial linear or angular velocity.

Note: The individual parts of the following question are intended to be independent from each other.

(a) A metal ring with radius $a$, mass $M$, and total resistance $R$ is oriented to lie on the $x-y$ plane. The ring moves along the $x$-direction and its center passes through the origin with velocity $v = v_0 \hat{i}$ at $t = 0$. The ring is immersed in a region of space with a magnetic field $B = B_0 \hat{x} = B_0 \hat{k}$. Assuming that $x_0 \ll a$, determine the distance the ring travels from the origin before it comes to rest.

(b) Consider a charged ring of radius $a$ with uniform charge density $\lambda$ centered about the $z$-axis at $z = \sqrt{3}a$ as shown in the figure.
   (i) What is the total charge $Q$ on the ring.
   (ii) Write down the volume charge density on the ring in terms of the total charge $Q$ and suitable Dirac-$\delta$ functions in spherical coordinates with respect to the origin $O$ as shown in the figure. Confirm that this correctly yields the total charge found in the previous part.
   (iii) Evaluate the nonzero components of the dipole $q_1m$, and the quadrupole $q_2m$ moments explicitly.

(c) An electric dipole $p$ is placed at a distance $d (d > R)$ pointing toward the center of a conducting sphere of radius $R$. Consider that the dipole has length $h$ and $|p| = qh$. Using the method of images
   (i) Determine the electric potential outside the sphere at a point $S$ with $|r| \gg h$, when the sphere is grounded.
   (ii) Determine the electric potential at the point $S$, when the sphere is electrically isolated and neutral.

(d) An electron of charge $e$ is released from rest and falls freely under the influence of gravity.
   (i) Using Larmor formula determine the total energy radiated away by the charge after it travels a distance of 10 meters.
   (ii) Determine the fraction of the potential energy lost in the form of radiation in the course of this motion. Do you think that we can safely neglect the energy loss due to radiation in this case? Explain why.

(e) A plane electromagnetic wave with wave vector $k = k\hat{n}$ is incident on a wall with incidence angle $\theta$ as shown in the figure. The wave is reflected with a reflection coefficient $R$. The energy momentum tensor for the incident wave can be written as

$$T^{\mu\nu} = \frac{u}{c^2 \omega^2} k^\mu k^\nu e^{-2i k \cdot x}$$

where $u$ stands for the energy density, $\omega$ for the frequency of the plane wave, and $k^\mu$ and $x^\mu$ are four vectors. The normal vector of the wall is $\hat{N}$ as depicted on the figure.
(i) Write down the relation between the frequency and wave number of the plane wave.

(ii) What is the force per unit area $F$ exerted on the wall in this process. Determine the component of this force normal to the wall. This is called the light pressure.

*Hint: Note that $F$ must be related to the Maxwell Stress tensor part of $T^{\mu\nu}$ and use the principle of superposition.*

**[EMT-2017-May] Q2: Average Hyperfine Interaction**

In the hydrogen atom, there is an interaction between the intrinsic magnetic moments of the proton and the electron, which is called as the hyperfine interaction. The interaction energy can be described as

$$H_{hf} = -\vec{\mu}_e \cdot \vec{B}_p(\vec{x})$$

where $\vec{\mu}_e$ is the electron’s magnetic moment, $\vec{x}$ is the position of the electron and $\vec{B}_p(\vec{x})$ is the magnetic field due to the magnetic moment of the proton, $\vec{\mu}_p$, at the position of the electron. For an electron in the 1s state, the average hyperfine energy can be expressed as

$$\langle H_{hf} \rangle = -\vec{\mu}_e \cdot \langle \vec{B}_p \rangle,$$

where $\langle \vec{B}_p \rangle$ is the average magnetic field seen by the electron,

$$\langle \vec{B}_p \rangle = \int \vec{B}_p(\vec{x}) |\psi(r)|^2 d^3x$$

and $\psi(r)$ is the 1s wavefunction, which depends only on the radial coordinate $r$. Below, you will compute $\langle \vec{B}_p \rangle$ and show that it depends only on the central value of the wavefunction, $|\psi(0)|^2$.

For simplicity, consider the proton’s magnetic moment $\vec{\mu}_p$ to be due to a current distribution $\vec{J}$ inside the proton. At the end, we will treat the proton as a point particle.

(a) Consider the sphere with radius $R$ which is centered at the proton. Compute the integral of $\vec{B}_p(\vec{x})$ on this sphere

$$\vec{c}_R = \int_{r<R} \vec{B}_p(\vec{x}) d^3x = R^2 \int \hat{n} \times A d\Omega$$

and show that

$$\vec{c}_R = \frac{2\mu_0}{3} \vec{\mu}_p.$$

Observe that this is independent of $R$.

*Hint: To perform this integral, carefully follow the steps given below:

1. Insert a general integral expression for $A$ in terms of the current density $J$ and rearrange the integrals.
2. Perform the integral $\int \frac{\hat{n} \cdot (\vec{\mu}_p)}{|x-x'|} d\Omega$ using $\hat{n} = \sqrt{\frac{2\pi}{3}} (-Y_{11}^* + Y_{1-1}^*) \hat{i} - \sqrt{\frac{2\pi}{3}} i (Y_{11}^* + Y_{1-1}^*) \hat{j} + \sqrt{\frac{4\pi}{3}} Y_{10}^* \hat{k}$ and the spherical harmonic expansion for $\frac{1}{|x-x'|}$.
3. Use the definition for magnetic moment $\vec{\mu}_p$ as an integral involving the current density to complete the calculation.*

(b) The magnetic field produced by the proton’s dipole moment at a distant location $\vec{x}$ can be expressed as

$$\vec{B}_p(\vec{x}) = \frac{\mu_0}{4\pi} \frac{3\hat{n} \cdot \vec{\mu}_p - \vec{\mu}_p}{|x|^3}. \quad (1)$$

Show that at a fixed radius $R \neq 0$

$$\oint_{r=R} \vec{B}_p(\vec{x}) d\Omega = 0.$$
Hint: Consider making use of the result \( \oint d\Omega \hat{n}_i \hat{n}_j = N \delta_{ij} \), where \( N \) is a constant that you need to determine.

Remark: Note that this result indicates that the average value of \( \bar{B}_p(\vec{x}) \) due to \( \vec{\mu}_p \) is vanishing on a sphere of radius \( R \). See the figure.

(c) Results in parts (a) and (b) indicate that the magnetic field produced by the proton must have a Dirac-\( \delta \) term that has to be added to Eq. (1). Make this correction to Eq. (1).

(d) Using the result of part (c), evaluate \( \langle \bar{B}_p \rangle \) and write down the value of the average hyperfine energy \( \langle H_{hf} \rangle \) in terms of \( \vec{\mu}_e \), \( \vec{\mu}_p \) and \( |\psi(0)|^2 \).

**[EM-2016-Nov]** Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

(a) A very long cylinder of radius \( R \) carries a uniform surface charge density \( \sigma \) and is set to rotate about its symmetry axis with angular speed \( \omega \).
   (i) Determine the electric field inside and outside the cylinder.
   (ii) Determine the magnetic field inside and outside the cylinder.

(b) Consider a sphere of radius \( R \) and with uniform polarization \( \vec{P} = P \hat{k} \).
   (i) Determine the bound surface charge density \( \sigma_b = \vec{P} \cdot \hat{r} \).
   (ii) Determine the electric potential inside and outside the sphere.
   (iii) What is the electric field inside the sphere.

(c) A conducting sphere of radius \( R \) carrying a total charge \( Q \) is located at a distance \( d \) from a very large grounded conducting sheet on the \( xy \) plane. If \( R \ll d \) then electric dipole approximation can be used where only the monopole and dipole terms are kept for the sphere. Using the method of images determine the electric potential at a point \( P \) above the \( xy \) plane with the dipole approximation.
   Hint: Note that the problem has azimuthal symmetry about the \( z \) axis. What does this imply for the dipole moment \( \vec{p} \)?

(d) In a reference frame \( K \), a stationary point charge \( q \) is located above a stationary infinite sheet of uniform surface charge density \( \sigma \). With respect to an observer in another inertial reference frame \( K' \) both the point charge and the infinite sheet are moving with a relativistic velocity \( \vec{v} = v \hat{y} \) in the \( y \) direction.
   (i) What is the electromagnetic force on the charge \( q \) in the \( K \)-frame?
   (ii) Determine the electromagnetic fields created by the sheet in the \( K' \)-frame.
   (iii) What is the total electromagnetic force on the point charge in the \( K' \)-frame?
Consider two equal point charges placed along the $z$ axis at $z = d$ and $z = -d$. Using the Maxwell stress tensor,

$$T_{ij} = \varepsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} |E|^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} |B|^2 \right),$$

determine the electric force between the two charges. Does your answer agree with the result you get from the Coulomb’s law? Hint: Integrate the Maxwell stress tensor over a surface enclosing the charge at $z = d$. Simply consider that this surface is formed from the entire $xy$-plane and a hemisphere above it with very large radius and neglect the integral over the latter. For simplicity, use polar coordinates while integrating over the $xy$-plane.

[EM-2016-Nov] Q2: Dipole in Motion

Consider an ideal electric dipole with moment $p$ moving with a non-relativistic velocity $v(t)$. The dipole moment $p$ has constant magnitude and a fixed direction in time. At a given time $t$ its position is given by the vector $r_0(t)$. Electric potential of this dipole may be expressed as

$$\Phi = -p \cdot \nabla \frac{1}{|x - r_0(t)|},$$
in Gaussian units.

(a) (i) Using the Poisson’s equation for $\Phi$, show that the charge density $\rho(x, t)$ of the dipole may be expressed as

$$\rho(x, t) = -p \cdot \nabla \delta(x - r_0(t)).$$

(ii) Show that the same charge density $\rho(x, t)$ can be obtained starting from the polarization $P = p \delta(x - r_0(t))$ and then computing the corresponding bound charge density.

(iii) What is the corresponding current density $J(x, t)$ of the dipole in motion?

(b) Using the current density $J(x, t)$ determined in the previous part, show that the dipole in motion has a magnetic moment $m$, which is given as

$$m = \frac{1}{2c} p \times v(t).$$

(c) Using the charge density determined in part (a), show that the dipole in motion has also an electric quadrupole moment $Q_{ij}$, which is given as

$$Q_{ij} = 3(r_0 p_j + r_0 p_i) - 2r_0(t) \cdot p \delta_{ij}. $$

Instantaneous power radiated per unit solid angle due to a magnetic dipole is given by

$$\frac{dP_m}{d\Omega} = \frac{1}{4\pi c^5} \left| \left( \frac{d^2m}{dt^2} \times \hat{n} \right) \times \hat{n} \right|^2,$$

where $\hat{n}$ is the unit vector from the origin along the direction at which the instantaneous power is calculated. Whereas, the total power radiated by a quadrupole is given as $P_Q = \frac{1}{180c^5} \sum_{ij} \left| \frac{d^3Q_{ij}}{dt^3} \right|^2$.

Let us now suppose that the dipole points in the $z$-direction, $p = p\hat{z}$, and rotates on a circle of radius $R$ on the $xy$-plane with angular velocity $\omega$. Thus we have $r_0(t) = R\cos\omega t \hat{x} + R\sin\omega t \hat{y}$.

(d) Compute $m$ and the non-vanishing components of $Q_{ij}$ for the motion of the dipole specified above.
(e) Determine both the instantaneous power per unit solid angle, \( \frac{dP_m}{d\Omega} \), and the total power \( P_m \) radiated due to the magnetic dipole moment found in part (d).

(f) Determine the total power \( P_Q \) radiated by the quadrupole moment found in part (d). Compare the frequency dependence of \( P_m \) and \( P_Q \).

**Hint 1:** Recall that \( \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi \delta(\mathbf{r} - \mathbf{r}') \).

**Hint 2:** In the calculation of \( \mathbf{m} \) and \( Q_{ij} \), use index notation and integration by parts and note for the latter that the total derivative terms give no contribution.

**Hint 3:** \( \int d\Omega \sin^2 \theta = \frac{8\pi}{3} \).

[EM-2016-May] Q1:
Answer the Following Questions.

*Note: The individual parts of the following question are intended to be independent from each other.*

(a) A very long circular cylinder of radius \( R \) has magnetization \( \mathbf{M} = \alpha r^2 \hat{\phi} \). Here \( r \) denotes the radial distance from the axis of the cylinder and \( \phi \) is the cylindrical polar coordinate. Determine the magnetic field inside (\( \mathbf{B}_{in} \)) and outside (\( \mathbf{B}_{out} \)) the cylinder.

**Hint:** Bound volume and surface current densities are given by \( J_b = \nabla \times M \), \( K_b = M \times \hat{n} \). Curl in cylindrical coordinates is given in the formula page.

(b) A radial current \( I_0 \) is flowing through an annular disk of inner radius \( R_1 \) and outer radius \( R_2 \) and thickness \( h \). A constant magnetic field \( \mathbf{B} \) is being applied perpendicular to the annular plane. Using Faraday’s law compute the motional EMF and indicate the direction of the induced current. Let \( n_e \) denote the number density of electrons with charge \( q \).

(c) A spherical shell of radius \( R \) carries a surface charge density \( \sigma(\theta, \phi) = \sigma_0 \sin \theta \cos \phi \). Using the general solution of the Laplace equation find the electric potential inside (\( \Phi_{in} \)) and outside (\( \Phi_{out} \)) of the spherical shell.

**Hint:** Consult the formula page for spherical harmonics.

(d) A charge \( q \) moves with velocity \( \mathbf{v} = v \hat{\mathbf{j}} \) in between and parallel to two long wires with linear charge densities \( \lambda \) and \(-\lambda\) as observed in the lab frame and shown in the figure.

(i) Determine the electromagnetic force acting on the charge in the lab frame.

(ii) Compute the electric and magnetic fields on the charge in its rest frame and find the electromagnetic force on it in this frame. How does your answer compare with that of part (i).

(e) Possible photon mass effects may be studied using the Proca equation \( \partial_\alpha \partial^\alpha A_\beta + \mu^2 A_\beta = 0 \) in the Lorenz gauge \( \partial_\alpha A^\alpha = 0 \), where \( A^\mu = (\Phi, \mathbf{A}) \) is the 4-vector potential. Poynting vector for Proca fields takes the form

\[
\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B} + \mu^2 \Phi \mathbf{A}) .
\]
Consider a plane Proca wave of unit amplitude $A = \hat{\varepsilon}_0 \cos(kz - \omega t)$, where $\hat{\varepsilon}_0$ is a polarization vector of unit magnitude, indicating either transverse or longitudinal polarizations. By first finding $\Phi$ in each case using the Lorenz gauge condition, determine the time averaged (over one period) energy fluxes for the following cases:

(i) Transversely polarized field.

*Hint: Take a linearly polarized field in one of the coordinate directions perpendicular to the direction of propagation.*

(ii) Longitudinally polarized field.

**[EM-2016-May] Q2: Dipole Meets a Conducting Sheet**

Consider an electric dipole $p$ placed at a distance $d$ from an grounded conducting infinite sheet placed on the $xz$-plane. At a given time the dipole makes an angle $\alpha$ with the horizontal as shown in the figure A.

(a) Determine the image charge configuration. Show it on the figure A.

(b) Find the total work done on the dipole as it rotates from $\alpha = 0$ to $\alpha = \frac{\pi}{2}$ configuration.

For the rest of the problem consider that the dipole is oriented in the vertical direction and oscillates with frequency $\omega$, that is we have $p = pe^{-i\omega t}\hat{k}$. See the figure B.

(c) Electric field due to the dipole only (i.e., excluding the effect of the conducting sheet), far away from the origin ($r \gg d$), at point $P$, can be approximated as $E_{\text{dip}} = k^2 \frac{e^{ikr - i\omega t}}{r} (\hat{n} \times p) \times \hat{n}$, where $r = r\hat{n}$ and $d = d\hat{\jmath}$. Assuming $d \ll r$, show that the total electric field at $P$ due to the dipole and conducting infinite sheet system is given by

$$E_T = k^2 p \frac{e^{ikr - i\omega t}}{r} \left( 1 - e^{i\delta} \right) \sin \theta \hat{\theta}.$$  

where $\theta$ is the angle between $p$ and $\hat{n}$ and $k = \frac{\omega}{c}$. Determine also the phase angle $\delta$.

(d) Determine the time averaged power radiated per unit solid angle, $\left\langle \frac{dP}{d\Omega} \right\rangle$, at the point $P$ by this system.

*Hint 1: Electric field due to a dipole $p$ is given by $E(x) = \frac{3\hat{n}(\hat{n} \cdot p) - p}{|x - x_0|^3}$ and the torque on a dipole in an electric field is $N = p \times E$.*

*Hint 2: Binomial expansion formula $\sqrt{1 + x} \approx 1 + \frac{1}{2} x$ for $x \ll 1$.*

*Hint 3: In the dipole approximation general expression for the magnetic field in the far zone is given by $B = \hat{n} \times E$.*

*Hint 4: Explicitly we have $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi} Re[r^2 \hat{n} \cdot E \times B^*]$.*

Figure A: For parts (a) and (b).

Figure B: For parts (c) and (d).
[EM-2015-Nov] Q1:
Answer the Following Questions. Note: The individual parts of the following question are intended to be independent from each other.

(a) A wire loop of radius \( R \) carrying a charge density \( \lambda \) is suspended horizontally on the \( xy \)-plane and is free to rotate. There is a uniform magnetic field \( \vec{B} = B_0 \hat{k} \) penetrating the circular region out to radius \( R/2 \) from the center of the wire loop. Determine the total angular momentum \( \vec{L} \) imparted to the wire loop as the magnetic field is switched off.

(b) Consider six identical electric dipoles \( \vec{p} \) placed at the center of each face of a cube of side length \( d \). The dipoles are placed parallel to each other but the direction they point is arbitrary. Determine the electric field at the center of the cube. Hint: Electric field of a point dipole is given by

\[
\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \frac{3\hat{n} \cdot \vec{p} - \vec{p}}{|\vec{x} - \vec{x}_0|^3}.
\]

(c) It is known that, just like the \( \vec{E} \) and \( \vec{B} \) fields are fitted in the electromagnetic field strength tensor \( F^{\mu\nu} \), electric dipole moment \( \vec{p} \) and magnetic moment \( \vec{\mu} \) may be fitted into a tensor \( \sigma^{\mu\nu} \) as

\[
\sigma^{\mu\nu} = \begin{pmatrix}
0 & -p_y & -p_z \\
p_x & 0 & -\mu_z \\
p_y & \mu_z & 0 \\
p_z & -\mu_y & \mu_x
\end{pmatrix}
\]

In analogy with the Lorentz invariants of \( F^{\mu\nu} \), determine the Lorentz invariants of \( \sigma^{\mu\nu} \) and express them in terms of \( \vec{p} \) and \( \vec{\mu} \).

In a given reference system, \( \vec{p} \) and \( \vec{\mu} \) are perpendicular and \( |\vec{p}| = 1 \), \( |\vec{\mu}| = 2 \). Is there a reference system in which \( \vec{\mu} \) vanishes?

(d) Two protons are separated by a distance \( d \) and they are moving parallel to one another with the velocity \( \vec{v} = \hat{v} \hat{i} \) with respect to a stationary observer in the \( K \)-frame. From the formula for the electric field of a moving charge, it is easily seen that the \( \vec{E} \)-field due to one of the protons at the instantaneous position of the other simply has the magnitude \( \gamma q/r^2 \). However, the force on each proton as measured by this observer is not \( qE = \gamma q^2/r^2 \). Find the force \( F \) on each proton in the reference frame \( K \), by finding the force in the proton rest frame and transforming it back to the \( K \)-frame. Evaluate the difference between \( F \) and \( qE \) in the \( K \)-frame and explain in detail how you can account for this difference.

(e) In a linear particle accelerator total instantaneous radiated power is given by

\[
P(t) = \frac{2e^2}{3c} \gamma^6 (\hat{\beta})^2,
\]
where \( \beta = \frac{\vec{v}}{c} \) and \( \gamma = \frac{1}{\sqrt{1 - \beta^2}} \) is the usual relativistic factor. Find an expression for \( P(t) \) in terms of the time derivative of the magnitude \( p \) of the momentum of the particle. Using the relativistic energy formula, show that rate of change of momentum is equal to the change in energy of the particle per unit distance and express \( P \) accordingly. Does \( P \) depend on \( \gamma \) and/or the energy of the particle? Energy gains in linear accelerators are usually less than 50 MeV/m. Given that \( \frac{\epsilon^2}{mc^2} = 2.82 \times 10^{-15} \) m and \( mc^2 = 0.511 \) MeV, conclude whether it is possible or not to ignore the radiation loss in linear accelerators?

**[EM-2015-Nov] Q2: Spinning the Dielectric Sphere**

A dielectric sphere with dielectric constant \( \varepsilon_r = \varepsilon/\varepsilon_0 \) and radius \( R \) is placed in a uniform external electric field \( \vec{E} = E_0\hat{k} \).

(a) Determine the electric fields \( \vec{E}_{in} \) and \( \vec{E}_{out} \) inside and outside the dielectric sphere.

(b) What is the polarization surface charge density \( \sigma_{pol} \) on the dielectric sphere.

Consider from now on that the dielectric sphere is set rotating with an angular velocity \( \omega \) about the \( z \)-axis.

(c) Determine the surface current density \( \vec{K} = \sigma \vec{\omega} \times \vec{r} \).

(d) Using \( \vec{K} = \vec{M}_{eff} \times \hat{n} \) argue that the effective magnetization \( \vec{M}_{eff} = \sigma_{pol} \omega R \hat{k} \).

(e) Compute the effective magnetic volume and surface charge densities \( \rho_M = -\nabla \cdot \vec{M}_{eff} \) and \( \sigma_M = \vec{M}_{eff} \cdot \hat{n} \).

(f) Obtain the magnetic scalar potentials \( \Phi_{in} \) and \( \Phi_{out} \) inside and outside the dielectric sphere.

Hint:

\[
\Phi = -\frac{1}{4\pi} \int_V \frac{\nabla' \cdot \vec{M}_{eff}'}{|\vec{x} - \vec{x}'|} d^3x' + \frac{1}{4\pi} \oint_S \frac{\vec{M}_{eff}' \cdot \hat{n}'}{|\vec{x} - \vec{x}'|} d^2x'
\]
**EM-2015-May** | Q1:

Answer the Following Questions. *Note: The individual parts of the following question are intended to be independent from each other.*

(a) An infinitely long circular cylinder has uniform magnetization $\vec{M} = M\hat{k}$ parallel to the axis of the cylinder. Determine the magnetic field inside ($\vec{B}_{\text{in}}$) and outside ($\vec{B}_{\text{out}}$) the cylinder.

(b) Consider a charge $-q$ at the origin and another charge $3q$ located on the $z$-axis at the point $(0,0,a)$. Determine the monopole, dipole and quadrupole moments of this configuration both in cartesian and spherical formulations. Write down the electric potential including the monopole, dipole and quadrupole contributions in either the cartesian or the spherical formulation.

(c) A particle with charge $q$ is confined to the $xy$ plane and is at rest at a point away from the origin. A magnetic field $\vec{B} = \Phi(t)\delta(x)\delta(y)\hat{z}$ is turned on, where $\Phi(t)$ increases at a constant rate starting from zero and the particle experiences a electric force $\vec{F}$. Determine this force and by computing its torque, show that $\vec{L} + \frac{q}{2\pi}\Phi\hat{k}$ is a constant of motion, where $\vec{L}$ is the mechanical angular momentum of the particle with respect to the origin.

(d) A certain magnetic dipole pulsar has period $T$ and a slow down rate $\left|\frac{dT}{dt}\right|$. The time-averaged total power radiated by this pulsar is $P = \frac{\mu_0}{12\pi} m^2 \omega^4$, where $\omega$ is the frequency, $\omega = \frac{2\pi}{T}$ and $m = |\vec{m}|$ is the magnitude of the magnetic dipole moment. Consider that, this radiated power is due to a decrease in the rotational kinetic energy of the pulsar. Find an expression for the maximum value of the magnetic field $|\vec{B}|$ on the surface of the pulsar in terms of $T$, $|\frac{dT}{dt}|$, mass $M$ and radius $R$ of the pulsar and the relevant constants.

*Hint: Consider the pulsar as a rigid sphere with moment of inertia $I = \frac{2}{5}MR^2$ and compute the time derivative of its rotational kinetic energy.*

The magnetic field of a dipole at a distance $r$ from the dipole is given as

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3}.\,$$

For what orientation of $\vec{m}$, $|\vec{B}|$ is maximized. Note also that near zone magnetic field can be taken as above.

(e) A charge is released in the presence of uniform electric and magnetic fields $\vec{E} = E\hat{k}$ and $\vec{B} = B\hat{i}$. Assuming that $E < B$, find a reference frame in which the the electric field vanishes, $\vec{E}' = 0$. Describe (or plot) the trajectory of the particle in this new reference frame.

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**EM-2015-May** | Q2: Dipole in Dielectric Sphere
A small electric dipole $\vec{p} = qd\hat{k}$ is embedded at the center of dielectric sphere of radius $R$ ($d \ll R$) and permittivity $\varepsilon$. The sphere is in vacuum of permittivity $\varepsilon_0$.

(a) Write down the electrostatic boundary conditions on the surface of the dielectric sphere.

(b) Determine the scalar potential inside ($\Phi_{in}$) and outside ($\Phi_{out}$) the dielectric sphere.

   Hint: Consider using the solutions of the Laplace equation combined with the fact that the scalar electric potential of the dipole is $\Phi_{dipole} = \frac{1}{4\pi\varepsilon} \frac{\vec{p} \cdot \vec{r}}{r^3}$.

(c) Determine the electric field inside ($\vec{E}_{in}$) and outside ($\vec{E}_{out}$) the dielectric sphere.

(d) Compute the bound surface charge density on sphere.

Note: The individual parts of the following question are intended to be independent from each other.

(a) Evaluate the complex integral
\[ \oint_C \frac{dz}{z^4 - 1} \]
over the contour \( C \) shown in the figure.

(b) Consider two point charges of unit strength \((q = 1)\) located on the \( z \)-axis at \( z = 1 \) and \( z = -1 \).

(i) Write down the electric potential \( \Phi(x) \) at any point on a unit sphere (except \( z = \pm 1 \)) and expand it in a series of Legendre polynomials in the form \( \Phi(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cos \theta) \). Determine the coefficients \( c_\ell \).

(ii) Evaluate the series \( \sum_{j=0}^{\infty} P_{2j}(0) \).

Hint: Recall that we have the formula
\[ \frac{1}{|x - x'|} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{r'^{\ell+1}} P_\ell(\cos \gamma), \]
where \( \cos \gamma = \hat{x} \cdot \hat{x}' = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \).

(c) Evaluate the integral
\[ J_m = \int_0^\infty \frac{x^{m-1}}{e^x - 1} \, dx, \quad m > 0 \]
by expanding an appropriate geometric series and using the definition of \( \Gamma(m) \) and \( \zeta(m) \).

Hint: The Riemann zeta function is \( \zeta(m) = \sum_{n=1}^{\infty} n^{-m} \).

(d) In the quantum mechanics exam one of the problems dealt with the Yukawa potential of the form
\[ V(r) = -\frac{e^{-\mu r}}{r} \]
where \( \mu > 0 \) is a constant. In quantum mechanics it becomes useful to know the Fourier transform of \( V(r) \) to compute the so called Schrödinger’s integral equation. Evaluate the three dimensional Fourier transform
\[ V(q) = \frac{1}{(2\pi)^\frac{3}{2}} \int V(r) e^{-iqr} \, d^3r. \]
(e) Consider the differential equation
\[ \nabla^2 \psi + \lambda^2 \psi = 0 , \quad \lambda > 0 , \]
subject to the homogenous boundary conditions \( \psi(0, y) = \psi(a, y) = 0 \) and \( \partial_y \psi(x, 0) = \partial_y \psi(x, a) = 0 \) in a square region of side length \( a \), as given in the figure.

(i) Determine the characteristic values and characteristic functions.

(ii) Construct the corresponding Green’s function \( G(x, x', y, y') \) in terms of the characteristic functions found in part (i).

Hint: Green’s function \( G(x, x', y, y') \) satisfies
\[ \nabla^2 G = -\delta(x - x') \delta(y - y') . \]

[MP-2017-May] Q2: Legendre Functions of the Second Kind

Consider the Legendre’s differential equation
\[ (1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0 , \quad \ell = 0, 1, 2, \cdots . \]

Legendre polynomials \( P_\ell(x) \) which may be given in terms of the Rodrigues representation as
\[ P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell , \quad x \in [-1, 1] . \]

are solutions to this differential equation in the interval \([-1, 1]\). In this problem we are going to explore solutions to this differential equation in the same interval, which are linearly independent from \( P_\ell(x) \) and thus named Legendre functions of the second kind.

(a) Consider Legendre’s differential equation for \( \ell = 0 \). By elementary methods show that the solution is
\[ \frac{1}{2} \ln \frac{1 + x}{1 - x} . \]

Note: No credit will be given if you simply verify it by substituting this expression into the differential equation.

(i) Using complex function techniques show that
\[ \frac{1}{2} \ln \frac{1 + x}{1 - x} = \tanh^{-1} x . \]

(b) Consider the functions \( Q_\ell(x) = P_\ell(x) \tanh^{-1} x + \Pi_\ell(x) \). Obtain the inhomogeneous differential equation satisfied by \( \Pi_\ell(x) \) and argue from your result that \( \Pi_\ell(x) \) must be a polynomial in \( x \). What is the degree of this polynomial.

(c) As a consequence of part b., \( \Pi_\ell(x) \) can be expanded in terms of the Legendre polynomials in the interval \([-1, 1]\) as
\[ \Pi_\ell(x) = \sum_{n=0}^{\ell-1} c_n P_n(x) . \]

Using the differential equation obtained in part (b), evaluate \( \Pi_2(x) \).

(d) An integral expression for \( Q_\ell(x) \) has the form
\[ Q_\ell(x) = \frac{1}{2^{\ell+1}} \int_{-1}^{1} dt \frac{(1 - t^2)^\ell}{(x - t)^{\ell+1}} . \]
Integrating this expression by parts \( \ell \) times show that another integral representation of \( Q_\ell(x) \) is

\[
Q_\ell(x) = \frac{1}{2} \int_{-1}^{1} dt \frac{P_\ell(t)}{x-t}
\]

(e) Using the integral form obtained in part d. show that

\[
Q_\ell(x) = P_\ell(x) \tanh^{-1} x + \Pi_\ell(x)
\]

**Hint:** Consider adding and subtracting an appropriate term to the numerator in the integrand of \( Q_\ell(x) \). Do not attempt to calculate \( \Pi_\ell(x) \), just leave it as an integral.

**[MP-2016-Nov] Q1: Answer the Following Questions.**

Note: The individual parts of the following question are intended to be independent from each other.

(a) Evaluate the complex integral

\[
\int_{C_i} \frac{dz}{z^2 - 1}
\]

over the contours \( C_i, i = 1, 2, 3, 4 \), shown in the diagram. In each case, first close the contour in a way you choose and then use residue methods.

(b) Consider the complex functions

\[
f(z) = \sinh \left( \frac{4}{z} \right), \quad g(z) = \frac{1}{(z-i)(z-2)}.
\]

(i) Determine the Laurent series expansion for these functions around the origin of the complex plane.

(ii) What is the residue of \( f(z) \) at its singularity.

(c) In a certain quantum mechanics problem the wave function at the origin takes the form

\[
\psi(0) = e^{-\pi a} \Gamma(1 + ia).
\]

Using the properties of the gamma function, show that

\[
\psi^*(0)\psi(0) = \frac{2\pi a}{e^{2\pi a} - 1}.
\]

**Hint:** Recall that \( \Gamma(z+1) = z\Gamma(z) \) and \( \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \).

(d) Following Green’s function appears in the discussion of forced, damped harmonic oscillator

\[
G(t) = \begin{cases} \frac{1}{\Omega} e^{-\gamma t} \sin(\Omega t) & t > 0, \\ 0 & t < 0. \end{cases}
\]

where \( \Omega \) and \( \gamma \) are positive constants. Find the Fourier transform \( g(\omega) \) of \( G(t) \). Where are the poles of \( g(\omega) \) located on the complex plane with \( \Re e(z) = \omega \).
(e) Consider the differential equation
\[ \nabla^2 \psi + \lambda^2 \psi = 0, \]
subject to the homogenous boundary conditions \( \partial_x \psi(0, y) = \partial_x \psi(a, y) = \partial_y \psi(x, 0) = \partial_y \psi(x, a) = 0 \) in a square region of side length \( a \), as in Figure 1 given below.

(i) Determine the characteristic values and characteristic functions.

(ii) Construct the corresponding Green’s function \( G(x, x', y, y') \) in terms of the characteristic functions found in part (i).

Hint: Green’s function \( G(x, x', y, y') \) satisfies
\[ \nabla^2 G = -\delta(x - x') \delta(y - y'). \]

**[MP-2016-Nov] Q2: Legendre Polynomials All the Way**

Legendre polynomials may be defined using the Rodrigues representation as
\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad x \in [-1, 1]. \]

(a) Using the Rodrigues representation, compute \( P_n(1) \) and \( P_n(-1) \).

(b) Using the Rodrigues representation and integration by parts prove the orthogonality of Legendre polynomials, that is, show that
\[ \int_{-1}^{1} P_n(x) P_m(x) \, dx = \frac{2}{2n+1} \delta_{nm}. \]

(c) Using the results of part a) and b), show that
\[ \delta(1 - x^2) = \frac{1}{2} \sum_{n=0}^{\infty} (4n + 1) P_{2n}(x). \]

(d) Deduce from the Rodrigues formula the contour integral representation for the Legendre polynomials as
\[ P_n(z) = \frac{1}{2^n} \frac{1}{2\pi i} \oint_C \frac{(t^2 - 1)^n}{(t - z)^{n+1}} \, dt, \]
where \( C \) encloses the point \( z \).

(e) For the contour \( C \) in the previous part take a circle of radius \( |\sqrt{z^2 - 1}| \) centered around the point \( z \) and show that a real integral representation of Legendre polynomials may be written as
\[ P_n(z) = \frac{1}{\pi} \int_{0}^{\pi} (z + \sqrt{z^2 - 1} \cos \alpha)^n \, d\alpha, \]

(f) Recall that the spherical harmonics \( Y_{lm}(\theta, \phi) \) and the associated Legendre polynomials \( P_l^m(x) \) may be expressed as
\[ Y_{lm}(\theta, \phi) = (-1)^m \frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} P_l^m(\cos \theta) e^{im\phi}, \quad P_l^m(x) = (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x). \]

Consider the angular momentum operator \( L = -i \mathbf{x} \times \nabla \) and recall that \( L_+ = L_1 + iL_2 \). Compute
\[ L_+ Y_{l0}(\theta, \phi), \]
and express your answer in terms of the spherical harmonics. Note that this is a familiar result in quantum mechanics.

*Hint: Do NOT attempt to express $L_\phi$ in spherical coordinates.*

**[MATH-2016-May] Q1: Answer the Following Questions.**

(a) Compute the integral
\[ \int_{-\infty}^{\infty} \frac{1}{x^4 + 16} \, dx. \]

(b) Express the function
\[ \tanh^{-1} z \]

in terms of complex logarithm function. Find the branch points of this function and give one suitable set of branch cuts. Evaluate $\tanh^{-1} \sqrt{3}i$.

(c) Compute the integral
\[ \int f(r) \nabla \cdot (\delta(r-a)\hat{r}) \, d^3x, \]

over the entire three-dimensional volume. Using the results of this integral verify that $\nabla \cdot (\delta(r-a)\hat{r}) = \frac{\pi^2}{r^2} \delta'(r-a)$, when considered as a part of an integrand in a three-dimensional integral. (Note that prime “′” denotes the derivative with respect to $r$).

(d) Find the Fourier transform $g(\omega)$ of the function
\[ f(x) = e^{-|x|}. \]

Using the inverse Fourier transform of your answer and a simple substitution infer the Fourier cosine transform of the function $g(x) = \frac{1}{1 + x^2}$.

(e) Consider the differential equation
\[ \nabla^2 \psi + \lambda^2 \psi = 0, \]

subject to the homogenous boundary conditions $\psi(0,y) = \psi(a,y) = \psi(x,0) = \psi(x,a) = 0$ in a square region of side length $a$, as in Figure 1 given below.

(i) Determine the characteristic values and characteristic functions.

(ii) Construct the corresponding Green’s function $G(x,x',y,y')$ in terms of the characteristic functions found in part (i).

*Hint: Green’s function $G(x,x',y,y')$ satisfies $\nabla^2 G = -\delta(x-x') \delta(y-y')$.*

![Figure 1: For the part (e)](image-url)
[MATH-2016-May] Q2: Laguerre Polynomials All the Way

Associated Laguerre polynomials have the integral representation

\[ L_n^k(x) = \frac{1}{2\pi i} \oint_C e^{-xz / (1-z)} \frac{z^{n+1}(1-z)^{k+1}}{z^{n+1}} dz , \]

where \( C \) is a closed counterclockwise contour enclosing the origin but \( not z = 1 \), as shown in the given figure.

(a) Using the given integral representation evaluate \( L_n^k(0) \).

(b) What is the generating function \( g(x, t) \) associated with \( L_n^k(x) \)?
   
   Hint: Recall that the generating function and \( L_n^k(x) \) should satisfy \( g(x, t) = \sum_{n=0}^{\infty} L_n^k(x)t^n \), with \(|t| < 1\).

(c) In the given integral form, make the transformation \( xz / (1-z) = s - x \) to a new integration variable \( s \) and obtain an alternative integral representation for \( L_n^k(x) \). Use this alternative integral form to obtain the Rodriguez formula

\[ L_n^k(x) = e^{x} e^{-x^n+k} d^n dx^n [e^{-x^n+k}] . \]

(d) Show that the expansion of the function \( x^p \) in terms of the associated Laguerre polynomials \( L_n^k(x) \) at fixed \( k \) gives

\[ x^p = (p + k)! p! \sum_{n=0}^{p} \frac{(-1)^n L_n^k(x)}{(n+k)! (p-n)!} , \quad 0 \leq x < \infty , \]

Hint 1: The associated Laguerre polynomials satisfy the orthogonality relation

\[ \int_0^{\infty} e^{-x} x L_n^k(x)L_m^k(x)dx = \frac{(n+k)!}{n!} \delta_{nm} . \]

Hint 2: Use the Rodriguez formula given in part (c).

(e) In quantum mechanics, wave function for the Hydrogen atom takes the form

\[ \psi_{n\ell m}(r, \theta, \phi) = \left[ \frac{\alpha^\ell (n - \ell - 1)!}{2^n (n + \ell)!} \right]^{1/2} e^{-\alpha r/2} L_{n-\ell-1}^{2\ell+1} (\alpha r) Y_{\ell m}(\theta, \phi) , \]

where \( \alpha = \frac{2}{na_0} \). Compute the expectation value

\[ \left< \frac{1}{r} \right> = \int d^3x \frac{1}{r} \psi_{n\ell m}^* \psi_{n\ell m} , \]

for the average displacement of the electron from the nucleus.

Hint: Recall that spherical harmonics fulfill the orthogonality relation \( \int d\Omega Y_{\ell m}^*(\theta, \phi) Y_{\ell' m'}(\theta, \phi) = \delta_{\ell\ell'} \delta_{mm'} \) and \( d^3x = r^2 dr d\Omega \).

[MATH-2015-Nov] Q1:

Answer the Following Questions.

(a) Obtain the Laurent series expansion of \( f(z) = \frac{1}{(z - 2)(z - 3i)} \) in the region \( 3 < |z| < 4 \) on the complex plane. Determine the residue of \( f(z) \) at \( z = 3i \) from this Laurent series.
(b) Using the Gamma function \( \Gamma(z) = \int_0^\infty e^{-t^z} dt \), evaluate the integrals
\[
\int_0^\infty e^{-x^6} x^{14} dx, \quad \int_0^\infty e^{-x^6} x^{6n+2} dx, \quad n : \text{positive integer}
\]

(c) Find the Fourier transform \( g(\omega) \) of the function
\[
f(x) = \begin{cases} 
1 & |x| < \frac{\pi}{2} \\
\delta(x^2 - \pi^2) & |x| > \frac{\pi}{2}
\end{cases}
\]
Express your result for \( g(\omega) \) as a real function of \( \omega \).

(d) Use the Rodrigues formula
\[
P_n(x) = \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n (x^2 - 1)^n
\]
of Legendre polynomials to evaluate the integral
\[
\int_{-1}^1 x^n P_n(x) dx.
\]

(e) Consider the one-dimensional wave equation given as
\[
\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0.
\]
Find the solution \( \psi(x, t) \) of this equation subject to the boundary conditions \( \left. \frac{\partial \psi(x, t)}{\partial x} \right|_{x=0} = 0 \) and \( \left. \frac{\partial \psi(x, t)}{\partial x} \right|_{x=L} = 0 \) and the initial conditions \( \psi(x, 0) = \cos \frac{4\pi x}{L} \) and \( \left. \frac{\partial \psi(x, t)}{\partial t} \right|_{t=0} = 0 \).

(f) Light always travels along a path which minimizes time. The speed of light in a medium is given by \( v = \frac{d\ell}{dt} = \frac{c}{n} \), where \( c \) is the constant speed of light in vacuum and \( n \) is the index of refraction of the medium. In two-dimensional \( xy \)-space, consider a light beam, initially at the origin along the \( x \) direction, traveling in a medium with a varying index of refraction given as \( n(x, y) = n_0(1 + \alpha y) \) where \( n_0 \) and \( \alpha \) are some positive constants. Drive the trajectory of the light beam, that is, find \( y = y(x) \).

[MATH-2015-Nov] Q2: Spherical Bessel Functions all the way

The differential equation for spherical Bessel functions reads
\[
x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} + [x^2 - n(n+1)]R = 0.
\]

(a) Verify by direct substitution that
\[
R_n(x) = \frac{1}{2\pi} \frac{(-2)^n n!}{x^{n+1}} \oint_{C_j} \frac{e^{-izx}}{(z+1)^{(n+1)}(z-1)^{(n+1)}} dz,
\]

\[\text{MATH-2015-Nov} \ Q2: \text{Spherical Bessel Functions all the way}\]

\[\text{Im } z \quad \text{Re } z \quad \text{Im } z \quad \text{Re } z\]

\[C_j \quad -1 \quad 1 \quad C_n\]

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R_n(x) = \frac{1}{2\pi} \frac{(-2)^n n!}{x^{n+1}} \oint_{C_j} \frac{e^{-izx}}{(z+1)^{(n+1)}(z-1)^{(n+1)}} dz,
\]
where $C$ is a closed contour enclosing the singularities $z = \pm 1$, is a solution of this differential equation.

(b) Spherical Bessel functions of the first and second kind are denoted as $j_n(x)$ and $n_n(x)$, respectively and in the integral representation given in part (a) they are specified by the contours $C_j$ and $C_n$ as shown in the figure above. Determine $j_0(x)$ and $n_0(x)$ by evaluating the integral representation $R_0(x)$ on these contours.

(c) Show by mathematical induction that spherical Bessel function of the first kind $j_n(x)$ can be given as

$$j_n(x) = (-1)^n x^n \left( \frac{1}{x} \frac{d}{dx} \right)^n \left( \frac{\sin x}{x} \right).$$

**Hint:** Recurrence relations are given as ($j'_n(x)$ is the derivative of $j_n(x)$)

$$j_{n-1}(x) + j_{n+1}(x) = \frac{2n + 1}{x} j_n(x),$$

$$n j_{n-1}(x) - (n + 1) j_{n+1}(x) = (2n + 1) j'_n(x).$$

(d) By substituting $R(kr) = Z(kr)/\sqrt{kr}$, show that $Z(kr)$ satisfies the Bessel’s equation. What is the order of $Z(kr)$ as a Bessel function.

(e) In quantum mechanical scattering processes, radial part of the Schrödinger equation takes the form of the spherical Bessel equation and the corresponding radial wave function in the asymptotic region $r \to \infty$ may be given as

$$\psi_k(r) = \frac{\sin (kr + \delta_0)}{kr},$$

where $k$ is the wave number and $\delta_0$ is the scattering phase shift. Evaluate the normalization integral

$$\int_0^\infty \psi_k(r) \psi_{k'}(r) r^2 dr.

**Hint:** Recall that the Fourier representation of the Dirac delta function is

$$\delta(k - k') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} dx.$$ First obtain a Fourier sine representation of $\delta(k - k').$

[MATH-2015-May] Q1:

Answer the Following Questions.

8 (a) Evaluate the integral

$$\oint_C \frac{e^{1z}}{(z-3)^2(z+5)} dz,$n

where $C$ is the circle $|z| = 4$ on the complex plane.

10 (b) Consider the one-dimensional wave equation given as

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0.$$

Find the solution $\psi(x,t)$ of this equation subject to the boundary conditions $\psi(0,t) = 0$ and $\psi(L,t) = 0$ and the initial conditions $\psi(x,0) = \sin \frac{3\pi x}{L}$ and $\frac{\partial}{\partial t} \psi(x,t) \bigg|_{t=0} = 0.$

8 (c) Find a particular solution of the equation

$$\nabla^2 \phi = \alpha \delta(x) \delta(y) \delta'(z).$$

where $\alpha$ is a constant and $\delta'(z) = \frac{\partial}{\partial z} \delta(z)$.

**Hint:** Use the fact that $\nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta(x) \delta(y) \delta(z).$
(d) Wave equation in one dimension leads to the Helmholtz equation

\[ \left( \frac{\partial^2}{\partial x^2} + k^2 \right) \psi(x) = f(x). \]

Suppose that the boundary conditions are such that the waves are traveling in the positive \( x \)-direction. Show that the Green’s function is given as \( G(x-x') = \frac{1}{2k} e^{ik|x-x'|} \) for \(-\infty < x, x' < \infty\), by solving

\[ \left( \frac{\partial^2}{\partial x^2} + k^2 \right) G(x-x') = -\delta(x-x'). \]

*Hint:* Start with solving the homogeneous equation \( \left( \frac{\partial^2}{\partial x^2} + \kappa^2 \right) \psi_H = 0 \).

(e) Evaluate the variation

\[ \delta \int_{-\infty}^{\infty} dx \psi^*(x) H \psi(x) = 0, \quad H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x), \]

subject to the constraint \( \int_{-\infty}^{\infty} dx \psi^*(x) \psi(x) = 1 \).

*Hint:* Determine a functional \( g \equiv g \left( \psi(x), \psi^*(x), \frac{\partial \psi}{\partial x}, \frac{\partial \psi^*}{\partial x}, x \right) \) and compute its Euler Lagrange equation with respect to \( \psi^*(x) \).

Is your answer a familiar equation in quantum mechanics? What is the physical meaning of the Lagrange multiplier in your answer?

[MATH-2015-May] Q2: Hermite Polynomials all the way

Consider the relation between the Hermite polynomials \( H_n(x) \) and their generating function \( g(x,t) = e^{-t^2 + 2tx} \).

\[ e^{-t^2 + 2tx} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n. \]

Treat \( t \) as a complex variable.

(a) Show that an integral representation of \( H_n(x) \) may be given as

\[ H_n(x) = \frac{(-1)^n n!}{2^n \pi^n} e^{x^2} \int_C \frac{e^{-t^2}}{(t-x)^{n+1}} dt, \]

where \( C \) is a contour enclosing the point \( x \).

(b) Using the integral representation given in part (a), compute \( H_{2n}(0) \) and \( H_{2n+1}(0) \).

(c) Normalized wave functions of a quantum mechanical harmonic oscillator are given as

\[ \psi_n(x) = 2^{-\frac{x^2}{2}} \pi^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x). \]

Consider the differential operator \( A = \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right) \). Show that \( A\psi_n(x) = c\psi_{n-1}(x) \) and determine the constant \( c \). Is your answer consistent with the annihilation operator \( a \) acting on the number eigenstate \( |n\rangle \)?

(d) Compute the integral

\[ \int_{-\infty}^{\infty} xe^{-x^2} H_n(x) H_m(x) dx, \]

The result of this integral is helpful in some quantum mechanical applications.

*Hint:* Recursion relations for Hermite polynomials are

\[ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad H'_n(x) = 2nH_{n-1}(x), \]

and the orthogonality relation reads

\[ \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n \sqrt{\pi} n! \delta_{nm}. \]