MIDDLE EAST TECHNICAL UNIVERSITY

Department of Physics

Sample of Past Qualifying Exam Questions

(Without Miscellaneous Questions)

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1. Quantum Mechanics

1.1 QM-2018-2

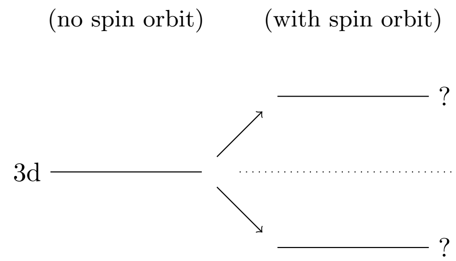
[QM-2018-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

14 (a) How do the 3d levels of the hydrogen atom split up under spin-orbit interaction? To explain in more detail: The Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{e^2}{r} + \frac{A}{r^3}\,\vec{\boldsymbol{L}}\cdot\vec{\boldsymbol{S}}\ ,$$

where A is some positive constant and the last term is treated as a perturbation. Without the perturbation, all the 3d states ($5 \times 2 = 10$) are degenerate.



With the perturbation, the energy levels split up into two or more levels. Do only enough calculations to answer the following. At the level of first-order perturbation theory,

(i) to how many levels do the 3d levels split?

(ii) What are the *quantum numbers* associated with each level?

(iii) What are the *degeneracies* of each level?

(iv) Which of these levels have their energy lowered and which have raised as a result of the perturbation?

12 (b) Consider a Dirac particle moving under the effect of a potential energy $V = V(\vec{\boldsymbol{r}})$. The Hamiltonian is $H_D = c\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\alpha}} + mc^2\beta + V$ and the corresponding eigenvalue equation is $H_D\psi = E\psi$. The non-relativistic Schrödinger equation can be obtained as an approximation of the Dirac equation by choosing appropriate large and small components of the Dirac spinor ψ. If you want to approximate non-relativistic positive-energy solutions (i.e., the energy satisfies $E \approx mc^2$), how would you choose the small and large components of ψ? After this, obtain the non-relativistic approximate version of the eigenvalue equation for the large component.

12 (c) The wavefunction of a particle, expressed in spherical coordinates is,

$$\psi(r,\theta,\phi) = N\, r\, e^{-\kappa r}\left(Y_1^1(\theta,\phi) + iY_1^0(\theta,\phi)\right)\ ,$$

where N is a normalization constant. Let $\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$ be the orbital angular momentum. Find the following expectation values.

(i) $\langle L_z \rangle$,

(ii) $\langle L_x \rangle$,

(iii) $\langle L_y \rangle$,

(iv) $\langle \boldsymbol{L}^2 \rangle$.

12 (d) For a single particle moving in 3D space, the x-translation operator is defined as

$$T(a) = \exp\left(-\frac{i}{\hbar}p_x a\right)$$

where p_x is the x component of the canonical-momentum operator. If $|\psi\rangle$ is a state of the particle, then $|\psi'\rangle \equiv T(a)|\psi\rangle$ corresponds to a state where the particle is displaced by a along the x direction.

(i) Using operator algebra, determine how x transforms under translations, i.e., find

$$T(a)^\dagger x T(a) \ .$$

Hint: Use the relation $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots$.

(ii) Using part (i), show that

$$\langle x \rangle_{\psi'} = \langle x \rangle_\psi + a$$

i.e., average position has moved by the same amount, a.

(iii) We say that the system's dynamics has *translational symmetry along x direction* if the Hamiltonian's expectation values do not change under translations, i.e.,

$$\langle H \rangle_{\psi'} = \langle H \rangle_\psi \quad \text{for all states } |\psi\rangle \text{ and for all } a \ .$$

As we all know, symmetry implies a conservation law. Show that, in the case of a translational symmetry, the x-component of momentum is conserved, i.e., $\langle p_x \rangle_t$ does not depend on time for any initial state.

[QM-2018-Nov] Q2: Radial Momentum

In solving quantum mechanics problems in spherical coordinates, one sometimes has to deal with the radial component of momentum as an operator. A simple minded candidate for the *radial momentum operator* that immediately comes to mind is the expression

$$\mathcal{P}_{\text{simple}} = \hat{\boldsymbol{r}} \cdot \vec{\boldsymbol{p}} = \frac{\hbar}{i} \hat{\boldsymbol{r}} \cdot \vec{\boldsymbol{\nabla}} = \frac{\hbar}{i} \frac{\partial}{\partial r} \ .$$

However, because of the curvilinear nature of the spherical coordinates, $\hat{\boldsymbol{r}}$ does not commute with $\vec{\boldsymbol{\nabla}}$, and $\mathcal{P}_{\text{simple}}$ is not the correct expression; it is not even hermitian. The correct expression for the radial momentum appears to be

$$p_r = \mathcal{P}_{\text{simple}} + \frac{\hbar}{ir} = \frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$$

Below, you will work on several properties of this operator.

Please note that all parts of the question are independent from each other. Any one of them can be solved without using the result of any other part but you are allowed to use information from other parts.

8 (a) Note that the hermitian conjugate of any operator $\hat{A}$ is defined by the relation

$$\int \psi^* (\hat{A}\phi)\, d^3\vec{\boldsymbol{x}} = \int (\hat{A}^\dagger \psi)^* \phi \, d^3\vec{\boldsymbol{x}} \ .$$

Use this relation to compute the hermitian conjugates of p_r and $\mathcal{P}_{\text{simple}}$. In this way show that p_r is hermitian ($p_r^\dagger = p_r$) while $\mathcal{P}_{\text{simple}}$ is not ($\mathcal{P}_{\text{simple}}^\dagger \neq \mathcal{P}_{\text{simple}}$). (Note: Impose the boundary condition that the wavefunctions do not diverge at $r = 0$.)

Hint: Use of spherical coordinates is critical in the proof. For this reason, at the first step, you need to express the integrals in spherical coordinates.

10 (b) A way to derive the expression for p_r is by taking the hermitian part of $\mathcal{P}_{\text{simple}}$, i.e.,

$$p_r \equiv \frac{\mathcal{P}_{\text{simple}} + \mathcal{P}_{\text{simple}}^\dagger}{2} = \frac{\hbar}{i} \frac{\hat{\boldsymbol{r}} \cdot \vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{r}}}{2} \ .$$

Show that this procedure gives the above defined p_r.

Hint: First compute the commutator $\vec{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}} \cdot \vec{\boldsymbol{\nabla}}$ carefully. Note that the ith Cartesian component of $\hat{\boldsymbol{r}}$ is given by $(\hat{\boldsymbol{r}})_i = x_i/r$. Also, note that $\partial r/\partial x_i = x_i/r$.

5 (c) Show that p_r can alternatively be expressed as

$$p_r = \frac{\hbar}{i} \frac{1}{r} \frac{\partial}{\partial r} r \ .$$

This expression is considered as a successive product of three operators. By using this expression, show also that

$$p_r^2 = -\hbar^2 \frac{1}{r} \frac{\partial^2}{\partial r^2} r \ = \ -\hbar^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \ .$$

15 (d) It is possible to define "eigenstates of radial momentum" by the eigenvalue equation

$$p_r \varphi_k = \hbar k \varphi_k \ ,$$

where k is a real number. Intuitively, we expect φ_k to describe probability waves which are radially propagating.

(i) Solve this equation for φ_k. Is the wavefunction normalizable?

(ii) Find the probability current density for these states,

$$\vec{\boldsymbol{J}} = \frac{\hbar}{2mi} \left(\varphi_k^* \vec{\boldsymbol{\nabla}} \varphi_k - \text{c.c.} \right) \ .$$

(iii) What is the direction of $\vec{\boldsymbol{J}}$? For positive k, is $\vec{\boldsymbol{J}}$ radially inward or radially outward? What if k is negative.

(iv) For free particles, the states φ_k are also energy eigenstates. As a result, their time-dependence can be expressed as

$$\varphi_k(\vec{\boldsymbol{x}}, t) = \varphi_k(\vec{\boldsymbol{x}}) e^{-iEt/\hbar}$$

where E is the corresponding energy. Based on this, show that the current density expression you have found above satisfies the *continuity equation* (except the origin).

It can be shown that the expression of the square of the momentum ($p^2 \equiv \vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{p}}$) has a particularly nice form in terms of p_r,

$$p^2 = p_r^2 + \frac{L^2}{r^2} \ ,$$

where $\vec{\boldsymbol{L}} = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{p}}$ is the orbital angular-momentum operator. This expression greatly simplifies the radial Schrödinger equation. Below you will see two applications.

5 (e) Consider Laplace's equation, $\nabla^2 F = 0$, which is frequently met in electromagnetism, for example. Suppose that you look for solutions of the form, $F = f(r) Y_\ell^m(\theta, \phi)$ where $f(r)$ is a function of the radial coordinate only. Using the fact that $p^2 = -\hbar^2 \nabla^2$, find all radial functions $f(r)$ such that F is a solution of the Laplace's equation.

7 (f) Consider a quantum particle moving under the effect of a spherically symmetric potential energy, $V = V(r)$, which has at most a Coulomb-like $1/r$ divergence at the origin, $r = 0$, i.e.,

$$r^2 V(r) \longrightarrow 0 \text{ as } r \longrightarrow 0.$$

Consider the expression

$$\psi = \psi(r, \theta, \phi) = R(r) Y_\ell^m(\theta, \phi)$$

as an energy eigenstate. The radial part of the wavefunction, $R(r)$, has the behavior $R(r) \sim r^\alpha$ around the origin. Find the exponent α. If you have found more than one solution for α, explain which exponent is physical.

1.2 QM-2018-1

[QM-2018-May] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

15 (a) The spin state of an electron has the Hamiltonian $H = \frac{\hbar\omega}{2}\sigma_z$. If the initial spin state at $t = 0$ is given by

$$\psi(0) = N \begin{bmatrix} 2 \\ 1 \end{bmatrix} ,$$

where N is a suitable normalization factor,

(i) compute the state $\psi(t)$ at time t,

(ii) find the expectation value of all components of the spin $\langle \vec{\boldsymbol{\sigma}} \rangle_t$ at time t,

(iii) and describe the spin motion qualitatively (using words).

10 (b) A hard-wall is placed at the equilibrium position ($x = 0$) of a typical harmonic oscillator. If particle mass is m and the oscillator frequency is ω, the potential energy in the presence of the hard wall is

$$U(x) = \begin{cases} +\infty & \text{for } x \le 0 , \\ \frac{1}{2}m\omega^2 x^2 & \text{for } x > 0 . \end{cases}$$

The Hamiltonian is $H = p^2/2m + U(x)$.

(i) Express the boundary conditions obeyed by the allowed wavefunctions $\psi(x)$ at $x = 0$ and $x = \infty$.

(ii) Let $\phi_n(x)$ ($n = 0, 1, 2, \dots$) denote the energy eigenfunctions of a normal harmonic oscillator *without* a hard wall. Which one of the functions $\phi_n(x)$ satisfy the boundary conditions of part (i)? Briefly state why these ϕ_n should also be eigenfunctions of H.

(iii) List all energy eigenvalues of H. Can you show that H does not have other energy eigenfunctions other than those in part (ii)?

15 (c) Consider a free Dirac particle with the Hamiltonian $H_D = c\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{\alpha}} + mc^2\beta$. We would like to find a 4×4 matrix F with the following property: For any eigenspinor ψ of H_D with eigenvalue E, (i.e., $H_D\psi = E\psi$), the spinor ψ' defined by

$$\psi'(\vec{\boldsymbol{r}}, t) \equiv F\,\psi^*(\vec{\boldsymbol{r}}, t) ,$$

where star denotes complex conjugation, is an eigenspinor with eigenvalue $-E$, i.e., $H_D\psi' = (-E)\psi'$.

(i) Determine whether F commutes or anti-commutes with each of the α_i and β matrices.

(ii) Using the (anti)commutation properties above, find F.
Hint: F is either one of the α_i, β matrices, or it is a product of a few of them. The (anti)commutation properties established in part (i) enables you to quickly identify what F is.

(iii) Find all of the zero momentum energy eigenstates of H_D and their corresponding energies. After that, check if the charge conjugation transformation really converts positive energy solutions to the corresponding negative energy solutions.

10 (d) Suppose that a harmonic oscillator is subjected to a constant uniform force F, under which the Hamiltonian becomes

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - Fx .$$

By using perturbation theory (treat the F dependent term as perturbation), find the leading order (i.e., the lowest non-vanishing) correction to the energy of the nth level.

[QM-2018-May] Q2: Trapped by a Penning trap

Penning trap is a device used for confining charged particles in a region of space by externally applied electromagnetic fields. Specially designed electrodes are used for creating a non-uniform electric field generated by a quadratic potential,

$$\phi = \phi_0(2z^2 - x^2 - y^2) \ ,$$

where $\phi_0 > 0$ is a constant. If only this E-field were present, a positively charged particle would do a bounded motion along the z-axis, but be unbounded along the directions in the xy-plane. This is in accordance with the *Earnshaw's theorem* which states that an electric field alone cannot provide stable equilibrium positions for charges. An additional sufficiently strong uniform magnetic field along the z-direction, $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$, would bind the motion along the directions in the xy plane as well. Hence, this combination of electric and magnetic fields can confine positively charged particles in a region of space. This is called a *Penning trap.*

Consider the motion of a positively charged particle, $e > 0$, with mass m under such electric and magnetic fields. The Hamiltonian can be expressed as

$$H = \frac{1}{2m}\left(\vec{\boldsymbol{p}} - e\vec{\boldsymbol{A}}\right)^2 + e\phi \ .$$

To keep the rotational symmetry around the z axis, the vector potential is chosen in the so-called symmetric gauge,

$$\vec{\boldsymbol{A}} = \frac{1}{2}\vec{\boldsymbol{B}} \times \vec{\boldsymbol{r}} \ .$$

Since this Hamiltonian is a quadratic function of the positions and momenta, it reduces to a sum of three harmonic oscillator Hamiltonians. In this problem, you will do this reduction and find the corresponding oscillator frequencies.

3 (a) Show that, for the gauge chosen above, we have $\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{A}} = \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{p}}$.

10 (b) Work out the Hamiltonian and show that it can be expressed as

$$H = H_{xy} + H_z$$

where

$$H_{xy} = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega_1^2(x^2 + y^2) - \frac{1}{2}\omega_2 L_z \ ,$$
$$H_z = \frac{p_z^2}{2m} + \frac{1}{2}m\omega_3^2 z^2 \ ,$$

where $L_z = (\vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}})_z = xp_y - yp_x$ is the z-component of orbital angular momentum. Find the parameters ω_i $(i = 1, 2, 3)$ in terms of the given quantities.

8 (c) For the motion on the xy-plane, the following complex valued coordinate and momentum is useful,

$$Q = x + iy \ ,$$
$$P = p_x + ip_y \ .$$

By using the position-momentum commutation relations ($[x_\mu, p_\nu] = i\hbar\delta_{\mu\nu}$), show that

$$[Q, P] = [Q^\dagger, P^\dagger] = 0,$$
$$[Q^\dagger, P] = [Q, P^\dagger] = 2i\hbar \ .$$

Note 1: Observe that all the other commutators vanish.
Note 2: As you will carry out many commutator calculations below, it is useful to remember a very simple rule: The above relations say that the only operator that Q does not commute with is $P^\dagger$ and their commutator is twice the usual position-momentum commutator. In other words, the momentum canonically conjugate to Q is $P^\dagger$. Similar comments also hold for the $Q^\dagger$ and P pair.

8 (d) Let's define the following operators.

$$a = \frac{1}{2}\sqrt{\frac{m\omega_1}{\hbar}}\left(Q + \frac{i}{m\omega_1}P\right) \ ,$$
$$b = \frac{1}{2}\sqrt{\frac{m\omega_1}{\hbar}}\left(Q^\dagger + \frac{i}{m\omega_1}P^\dagger\right) \ ,$$

Show that, both a and b are ladder operators which correspond to independent degrees of freedom. In other words, show that

$$[a, a^\dagger] = [b, b^\dagger] = 1 \ ,$$
$$[a, b] = [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0.$$

8 (e) Show that L_z and H_{xy} can be expressed as

$$\begin{aligned} L_z &= \frac{i}{2}\left(QP^\dagger - Q^\dagger P\right) \ , \\ &= \hbar(a^\dagger a - b^\dagger b) \ , \\ H_{xy} &= \frac{P^\dagger P}{2m} + \frac{1}{2}m\omega_1^2 Q^\dagger Q - \frac{\omega_2}{2}L_z \ , \\ &= \hbar\omega_a\, a^\dagger a + \hbar\omega_b\, b^\dagger b + \epsilon_0 \ . \end{aligned}$$

Find the frequencies ω_a, ω_b and the zero-point energy ϵ_0 in terms of ω_1 and ω_2.

3 (f) How strong should the magnetic field $\vec{\boldsymbol{B}}$ be so that it guarantees the positivity of all frequencies? (*Note: This is same as the condition that the Penning trap really traps the particles.*)

10 (g) Briefly describe the energy eigenvalues of $H = H_{xy} + H_z$. What are the quantum numbers? What are the corresponding energies? What is the value of the angular momentum for these levels?

1.3 QM-2017-2

[QM-2017-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

12 (a) Consider a normalized spinor

$$\psi = \begin{bmatrix} a \\ b \end{bmatrix}$$

where $|a|^2 + |b|^2 = 1$.

(i) Compute the expectation values $\langle \sigma_x \rangle$, $\langle \sigma_y \rangle$, $\langle \sigma_z \rangle$, work out the expression $\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2$ and show that $\langle \vec{\boldsymbol{\sigma}} \rangle$ is a unit vector.

(ii) Consider the particular state where $a = 2/\sqrt{5}$ and $b = i/\sqrt{5}$. This state is a spin-up state along some direction $\hat{\boldsymbol{n}}$. In other words, ψ is an eigenstate of $\sigma_n = \vec{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{n}}$, the component of spin along $\hat{\boldsymbol{n}}$, with eigenvalue $+1$ ($\sigma_n \psi = \psi$). Find $\hat{\boldsymbol{n}}$.
Hint: In this state $\langle \sigma_n \rangle = 1$.

10 (b) Consider a harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

in an arbitrary initial state at $t = 0$.

(i) Compute $\frac{d}{dt}\langle x \rangle_t$ and $\frac{d}{dt}\langle p \rangle_t$ where $\langle \cdots \rangle_t$ represents the expectation value evaluated at time t.

(ii) The equations above can be solved easily by using the alternative expression $\alpha_t = \langle x \rangle_t + i\langle p \rangle_t / m\omega$. Find the differential equation satisfied by α_t (i.e., express $\frac{d}{dt}\alpha_t$ in terms of α_t). Solve it and express α_t in terms of its initial value.

(iii) Using the solution for α_t found above, and the corresponding expression for α_t^*, find compact expressions of $\langle x \rangle_t$ and $\langle p \rangle_t$ in terms of their initial values $\langle x \rangle_0$ and $\langle p \rangle_0$.

8 (c) Consider a particle in a one-dimensional box with length L (say $0 < x < L$). Even though this problem is exactly solvable, it is a good exercise to estimate the ground state energy by a suitable trial wavefunction. The wavefunction

$$\psi(x) = \begin{cases} Nx(L-x) & \text{if } 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases}$$

where N is a normalization constant, satisfies the boundary conditions and looks similar to the real ground-state wavefunction. Compute the expectation value of the Hamiltonian, $\langle H \rangle$ as an estimate of the real ground-state energy. What does the variational theory tell you about your estimate? Discuss and compare with the exact ground state energy.

10 (d) Three spin 1/2 atoms interact with each other through their spins. The Hamiltonian of the interaction can be expressed as

$$H = J(\vec{\boldsymbol{S}}_1 \cdot \vec{\boldsymbol{S}}_2 + \vec{\boldsymbol{S}}_2 \cdot \vec{\boldsymbol{S}}_3 + \vec{\boldsymbol{S}}_3 \cdot \vec{\boldsymbol{S}}_1)$$

where $\vec{\boldsymbol{S}}_i$ is the spin of the ith atom. Find the energy levels of the system. What are the degeneracies?
Hint: Try to express the Hamiltonian in terms of the total spin.

10 (e) Space inversion (or parity) is the transformation $(x^\mu) = (ct, x, y, x) \longrightarrow (x'^\mu) = (ct, -x, -y, -z)$. The Dirac spinor transforms under parity as follows,

$$\psi(x) \longrightarrow \psi'(x') = A\psi(x) \ ,$$

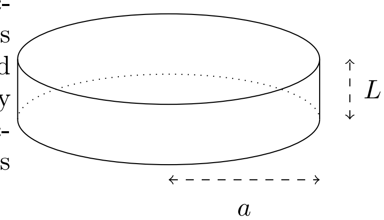
where A is some 4×4 matrix. We say that *the Dirac equation has inversion symmetry*, if for every solution $\psi(x)$ of the Dirac equation, $(i\gamma^\mu \partial_\mu + m)\psi = 0$, the transformed spinor $\psi'(x')$ also satisfies the same equation: $(i\gamma^\mu \partial'_\mu + m)\psi' = 0$.

(i) Find all conditions that the matrix A should satisfy so that the Dirac equation has inversion symmetry.

(ii) Show that the above conditions can be satisfied by a particular matrix. What is this matrix?

[QM-2017-Nov] Q2: Zeeman Effect in Cylindrical Quantum Dots

A *quantum dot* is a small structure inside semiconductors, in which electrons can be captured. Consider a cylindrical quantum dot with radius a and length L. In cylindrical coordinates, the dot region is described by the conditions $0 < z < L$ and $r < a$. Let's idealize the structure by assuming that the potential energy is constant inside the dot and the electrons cannot escape outside. As a result, the potential energy function is expressed by using the cylindrical coordinates as



$$U(\vec{\boldsymbol{r}}) = U(\rho, \varphi, z) = \begin{cases} 0 & \text{inside (if } 0 < z < L \text{ and } r < a) \\ \infty & \text{outside} \end{cases}$$

Consider a situation where a single electron is captured inside. Essentially, this is a 3-dimensional version of the *particle in a box* problem.

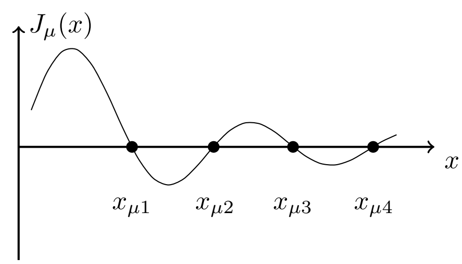
19 (a) Solve for the energy eigenfunctions $\Psi(\rho, \varphi, z)$ of the electron by using the method of separation of variables in cylindrical coordinates (i.e., assume that $\Psi(\rho, \varphi, z) = R(\rho)F(\varphi)G(z)$). Also answer the following questions:

(i) What are the boundary conditions satisfied by the energy eigenfunctions $\Psi(\rho, \varphi, z)$?

(ii) Show that the energy eigenfunctions are given as

$$E_{n\mu k} = \frac{\hbar^2}{2m}\left(\frac{n^2\pi^2}{L^2} + \frac{x_{\mu k}^2}{a^2}\right)$$

where m is the mass of the electron, n, μ and k are some quantum numbers, and $x_{\mu k}$ represents the kth zero of the Bessel function $J_\mu(x)$.



(iii) What are the corresponding normalized wavefunctions $\Psi_{n\mu k}(\rho, \varphi, z)$?

(iv) What are the possible values of the quantum numbers (i.e., for which values of n, μ and k we get distinct solutions for wavefunctions)?

(v) What is the degeneracy of each level? (Ignore the spin degeneracy.)

Hints: Bessel equation (for $J_\mu(t)$): $t^2y'' + ty' + (t^2 - \mu^2)y = 0$, for integer μ: $J_{-\mu}(t) = (-1)^\mu J_\mu(t)$, and $\int_0^u tJ_\mu^2(t)dt = \frac{1}{2}(u^2 - \mu^2)J_\mu^2(u) + \frac{1}{2}u^2(J'_\mu(u))^2$.

7 (b) In a particular experiment, light sent along the z direction is used for inducing transitions in the quantum dot. The electron and the light couple through the electric field of the light, which is on the xy plane. Therefore, by the dipole selection rule, this light can be absorbed (or emitted) in an $n\mu k \longrightarrow n'\mu'k'$ transition if the corresponding matrix elements of either x or y are non-zero:

$$\langle \Psi_{n'\mu'k'}|x|\Psi_{n\mu k}\rangle \neq 0 \quad \text{or} \quad \langle \Psi_{n'\mu'k'}|y|\Psi_{n\mu k}\rangle \neq 0\ .$$

Show that the selection rules for such transitions are $\mu' = \mu \pm 1$, $n' = n$.

4 (c) Suppose that the quantum dot has a thickness which is *much smaller* compared with its radius ($L \ll a$). Consider the lowest four energies: With the help of the table of zeros of Bessel functions given below, identify all quantum numbers associated with these levels. How many quantum states are associated with these levels? Which transitions are possible?

10 (d) Suppose that the quantum dot is placed in a magnetic field along the z-direction. Use the symmetric gauge given by

$$\vec{\boldsymbol{A}} = \frac{1}{2}\vec{\boldsymbol{B}} \times \vec{\boldsymbol{r}} = \frac{Bx}{2}\hat{\boldsymbol{y}} - \frac{By}{2}\hat{\boldsymbol{x}} \ .$$

Ignore the spin degree of freedom. Express the first order changes in the energies of all states $\Psi_{n\mu k}$.

Hint: $L_z = xp_y - yp_x = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$.

10 (e) Consider the absorption lines observed in the experiment described in part (b). For simplicity, consider only the lines due to transitions between the states you have discussed in part (c). How do the absorption line frequencies change by the magnetic field? Are all transition lines split by the magnetic field? To how many lines do they split?

	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$
$x_{\mu 1}$	2.4048	3.8317	5.1356	6.3802	7.5883
$x_{\mu 2}$	5.5201	7.0156	8.4172	9.7610	11.0647
$x_{\mu 3}$	8.6537	10.1735	11.6198	13.0152	14.3725
$x_{\mu 4}$	11.7915	13.3237	14.7960	16.2235	17.6160

First few zeros $x_{\mu k}$ of the first few Bessel functions, $J_\mu(x)$.

1.4 QM-2017-1

[QM-2017-May] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) The hyperfine interaction is the name given to the magnetostatic interaction of the internal magnetic moments of the electron and the proton in the hydrogen atom. For the case when the electron is in the 1s level, the spin states of the proton and electron are governed by the Hamiltonian

$$H_{\mathrm{hf}} = A\vec{\boldsymbol{S}}_e \cdot \vec{\boldsymbol{S}}_p \ ,$$

where A is a positive constant. Consider only the spins states.

(i) Does this Hamiltonian has rotational symmetry? (Explain in words, i.e., do not try to compute a commutator.) If so, which observable is conserved? (Words only.)

(ii) Find all energy eigenstates of H_{hf}. Find also the energy eigenvalues and their degeneracies. Which level is the ground state?

(iii) Find the frequency of photons emitted due to transitions between these levels. *Note: this is the famous 21 cm line that comes from atomic hydrogen in interstellar nebulae.*

10 (b) For a harmonic oscillator in 1D, carefully sketch

(i) the wavefunctions and

(ii) the probability density

as a function of position for the lowest three levels.

10 (c) The Dirac Hamiltonian is given by $H_D = c\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{p}} + mc^2\beta$.

(i) Compute the time derivative

$$\frac{d}{dt}\langle x_i \rangle$$

for an arbitrary state. Using this, identify the velocity operator.

(ii) Is it possible to measure two different components of the velocity at the same time?

(iii) What are the eigenvalues of the x-component of the velocity operator?

10 (d) Consider the following Hamiltonian

$$H = \frac{1}{2m}\left(\vec{\boldsymbol{p}} + \frac{eB}{c}\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}}\right)^2 - \frac{e^2}{r} + \frac{eB}{2mc}S_z + e\mathcal{E}x \ .$$

(i) Describe the physics that this Hamiltonian describes (i.e., what is the system that this is applied? The system is subject to which fields, etc.).

(ii) Which terms should vanish if H is inversion (parity) symmetric?

(iii) Which terms should vanish if H has time reversal symmetry?

(iv) Which terms should vanish if H has rotational symmetry?

5 (e) Consider a particle in one-dimension. Show that all expectation values of the operator $\hat{x}\hat{p}$ has an imaginary part. What is this imaginary part?
Hint: For a complex number z, the imaginary part is given by $\mathrm{Im}\, z = (z - z^*)/2i$. Try to simplify $\mathrm{Im}\,\langle \hat{x}\hat{p} \rangle$.

5 (f) For a free particle in one dimension, find the Heisenberg picture operators $p_H(t)$ and $x_H(t)$. Check the values of these expressions for $t = 0$.

[QM-2017-May] Q2: Searching Photon Mass in Hydrogen Atom

Even though we usually treat photons as massless particles, it is possible that they have an extremely small, but non-zero mass, m_γ, which we haven't noticed up to now. Observational data of various kinds show no evidence of a non-zero value for m_γ. On the other hand, we cannot prove that the mass is exactly zero by experimental means as the experiments always have errors and the physical effects of the photon mass can possibly be smaller than the errors. The best we can do is to place experimental *upper bounds* on the value of m_γ. Thus, if m_γ is non-zero, it must be extremely small. In this problem, the photon mass effects on the hydrogen atom will be investigated, which will then be used to place an upper bound on m_γ.

The existence of the photon mass changes the Coulomb interaction energy between the electron and the proton to the Yukawa form,

$$V_{\text{Coulomb}} = -\frac{e^2}{r} \quad \longrightarrow \quad V_{\text{Yukawa}} = -\frac{e^2}{r} e^{-r/\lambda} \ ,$$

where $\lambda = \hbar / m_\gamma c$ is the "reduced" Compton wavelength corresponding to the photon mass. The smallness of the photon mass translates into the largeness of the Compton wavelength as compared to the Bohr radius, i.e., $\lambda \gg a_0$.

5 (a) First of all, sketch the Coulomb and Yukawa potentials on the same graph and give one physical difference between them.

5 (b) Below, you will compute the effect of the Yukawa form on the Hydrogen atom by using first-order perturbation theory. For this purpose, expand the exponential term in the Yukawa potential (by assuming λ is large) and show that the Hamiltonian can be approximated as

$$H \approx \frac{p^2}{2\mu} - \frac{e^2}{r} + c_1 + c_2 r = H_0 + H'$$

where c_1 and c_2 are some constants and $H' = c_1 + c_2 r$ can be treated as a perturbation. What are c_1 and c_2?

10 (c) The bound-state wavefunctions of the hydrogen atom are given by $\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r) Y_\ell^m(\theta,\phi)$. For the special case of $\ell = n-1$ (i.e., 1s, 2p, 3d, 4f, … states), the radial wavefunction have the simple form

$$R_{n,n-1} = N_n r^\ell e^{-r/n\,a_0} \qquad (\ell = n-1).$$

Starting from the normalization relation for $\psi_{n\ell m}$, derive the corresponding relation for the radial wavefunctions $R_{n\ell}$. Show that the normalization constant for the $\ell = n-1$ states are given as

$$N_n = \left(\frac{2}{na_0}\right)^{n+\frac{1}{2}} \frac{1}{\sqrt{(2n)!}} \ .$$

15 (d) Using perturbation theory, compute the corrections to the energies of the states with $\ell = n-1$. (Note: The perturbed energies should be computed correctly in $1/\lambda$ and $1/\lambda^2$ orders.)

5 (e) First consider the binding energy of the hydrogen atom. The best experimental value of the ground state (obtained from binding energy measurements) is

$$E_{1s} = -13.598\,434\,48 \pm 9 \times 10^{-8}\,\text{eV} \ .$$

This value is consistent with the theoretically obtained value without taking the photon-mass effects into account. For this reason, if the photons really have mass, then the perturbation caused by the photon-mass effects must be smaller than the experimental error given above.

Using this observation, find a bound on the λ/a_0 ratio. Re-express this as a bound on the photon rest-energy, $m_\gamma c^2$ in eV units. (Note: $\hbar c/a_0 \approx 3700\,\text{eV}$.)

5 (f) Photon-mass effects also cause small changes to the transition frequencies. Using the results in part (d), find the lowest order change in the 2p⟶1s transition frequecy.

5

(g) The frequencies of 2p $\longrightarrow$ 1s transition lines have been measured with a fractional error of 10^{-11}. The measurements are consistent with the theoretical calculations that exclude the photon-mass effects. Again, if the photon mass exists, then its effect on the transition frequency should be smaller than the experimental error.

Use this fact to find a bound on λ/a_0. Translate this to a bound on the rest energy of the photon.

You may need the following information:

Gamma function integral: $$\int_0^\infty u^n e^{-su} du = \frac{n!}{s^{n+1}} \ .$$

Energy levels of the unperturbed H atom: $$E_n = -\frac{e^2}{2a_0}\frac{1}{n^2} \ .$$

1.5 QM-2016-2

[QM-2016-Nov] Q1: Answer the Following Questions.

8 (a) Consider a Dirac particle with mass m.

(i) Compute the anti-commutator $\{\alpha_i, H_D\} = \alpha_i H_D + H_D \alpha_i$ as an operator expression.

(ii) Consider an energy eigenstate with energy E. Starting with the expression $\langle\{\alpha_i, H_D\}\rangle$ show that

$$\langle p_i \rangle = \frac{E}{c} \langle \alpha_i \rangle \ .$$

Interpret this result in terms of the momentum-velocity relation of relativistic particles.

10 (b) A particle in 1D that moves under the effect of a uniform force F has the Hamiltonian,

$$H = \frac{p^2}{2m} - Fx \ .$$

Write down the Schrödinger equation for the *momentum-space wavefunction*, $\phi(p')$ for energy E, and solve it. After that, express the position-space wavefunction $\psi(x')$ as an integral expression. *Note: Do not try to evaluate the integral (the integral will be an Airy function).*

10 (c) Consider the nth energy level of a particle in a 1D box of length L. The force applied by the particle on one of the walls (say the wall on the right) can be computed in two different ways.

(i) We can consider the wall as an object with position coordinate L and "potential energy" given by E_n, the total energy of the system. In that case, the force on the wall is given by

$$F_n = -\frac{\partial E_n}{\partial L} \ .$$

Find the force F_n.

(ii) Alternatively, we can treat the particle semiclassically, i.e., we consider it as a classical particle with energy E_n, going back and forth in the box, periodically bumping and reflecting from the walls. Using this approach, compute the speed v_n of the particle, the impulse delivered on the right wall for each collision and the frequency of collisions with the right wall. Combining these compute the average force applied on the right wall.
Do not forget to compare your result with what you have found in part (i).

6 (d) Consider the non-relativistic Hydrogen atom with spin-orbit interaction. Which observables are conserved? Which quantum numbers (associated to which observables) can be used to label the energy eigenstates?

6 (e) Spin degrees of two spin-1 atoms are interacting with the Hamiltonian

$$H = -A\vec{\boldsymbol{S}}_1 \cdot \vec{\boldsymbol{S}}_2 \ ,$$

where $A > 0$ is a constant and $\vec{\boldsymbol{S}}_i$ is the spin of the ith atom. Find the energy eigenvalues and their degeneracies?

6 (f) An electron in a Hydrogen atom is in the state

$$|\Psi\rangle = \frac{1}{\sqrt{3}} \left(\psi_{100} + \psi_{210} + \psi_{211}\right) |\uparrow\rangle \ ,$$

where $\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r) Y_\ell^m(\theta,\phi)$. Find

(i) $\langle L^2 \rangle$,

(ii) $\langle L_x \rangle$.

4 (g) Compute the ladder operator in the Heisenberg picture, $a_H(t)$ (Here a is the ladder operator for 1D harmonic oscillator).

[QM-2016-Nov] Q2: Dipole Transitions for an anharmonic oscillator

The *dipole selection rule* states that a transition between two energy levels, $|\psi_n\rangle$ and $|\psi_m\rangle$, is possible if the position matrix element between these levels, $x_{nm} = \langle\psi_n|x|\psi_m\rangle$, is non-zero. The transition takes place with the emission or absorption of a single photon and the same condition holds for the forward, $\psi_n \to \psi_m$, as well as the backward transitions, $\psi_m \to \psi_n$. (In 3D, there is a transition if at least one of x_{nm}, y_{nm} and z_{nm} is nonzero.)

5 (a) Consider a 1D motion with the Hamiltonian

$$H = \frac{p^2}{2M} + V(x) \ .$$

Let $|\psi_n\rangle$ be the energy eigenstates with eigenvalues E_n. Compute the commutator $[H, x]$. After that, by simplifying the expression $\langle\psi_n|[H, x]|\psi_m\rangle$, deduce that the momentum matrix elements are given by

$$p_{nm} = \frac{iM(E_n - E_m)}{\hbar} x_{nm} \ .$$

Note: By using this relation, we conclude that the dipole selection rule can also be expressed in terms of the momentum matrix elements.

5 (b) If the potential energy $V(x)$ has inversion symmetry, $V(x) = V(-x)$, then $|\psi_n\rangle$ are parity eigenstates. Which transitions are forbidden by the inversion symmetry? Why?

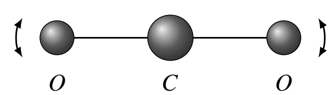
5 (c) Consider a harmonic oscillator in 1D,

$$H_0 = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 x^2 \ .$$

Using the dipole selection rule, determine the allowed transitions. In other words, if the system is at the nth energy level, to which levels it can make a single photon transition?

We usually model molecular oscillations by a harmonic oscillator, but some anharmonic effects are always present. Depending on presence of the inversion symmetry ($x \leftrightarrow -x$), we can model the anharmonic effects in two different ways:

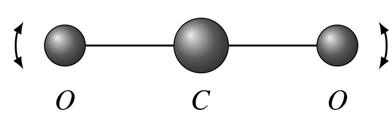
(S) Some oscillators have inversion symmetry, for example the bending mode of the CO_2 molecule. In this case, we have $V(x) = V(-x)$ and the simplest Hamiltonian that includes anharmonic effects can be written as



$$H_S = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 x^2 + \lambda x^4 \ ,$$

where λ is a small perturbation parameter.

(A) Some oscillators do not have inversion symmetry, for which the bending mode of the H_2O molecule is a good example. In this case, $V(x)$ can have a third order term and the Hamiltonian appears as



$$H_A = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 x^2 + \alpha x^3 \ ,$$

where α is also small.

Because of anharmonic terms, some of the forbidden transitions of the ideal harmonic oscillator may become allowed. In the remainder of this problem, you are going to identify these transitions.

15 (d) Consider the inversion symmetric case and the corresponding Hamiltonian H_S.

(i) By using perturbation theory, explicitly compute $x_{n,n+5} = \langle\psi_n|x|\psi_{n+5}\rangle$ up to (and including) first order in λ. Does this matrix element vanish?
Note: Take $x = x_0(a + a^\dagger)$ and do not express the constant x_0 in terms of the given parameters. Your main purpose in here is to identify vanishing and non-vanishing matrix elements.

(ii) Identify all transitions $n \to m$ that are made possible by the anharmonic terms to order λ^1. *Note: The explicit computation of x_{nm} is complicated and not needed for the rest of the problem. You only need to identify non-zero matrix elements in here; do not try to compute them all.*

(iii) Suppose that you are looking at the emission spectrum of a hot gas with molecules that possess such an anharmonic mode. Sketch a plot of the intensity of emitted light vs frequency graph of such a gas.

15 (e) Now, consider H_A, i.e., the oscillators that lack inversion symmetry.

(i) By using perturbation theory, explicitly compute $x_{n,n+4} = \langle\psi_n|x|\psi_{n+4}\rangle$ up to (and including) first order in α.

(ii) Identify all transitions $n \to m$ that are made possible by the anharmonic terms to order α^1.

(iii) Sketch a plot of the intensity of emitted light vs frequency graph of a gas having molecules with such oscillators.

5 (f) Water molecule H_2O has three oscillation modes with frequencies given below.

Modes	Frequency (THz)	Wavelength (μm)
mode-1	109.64	2.7344
mode-2	47.787	6.2708
mode-3	112.60	2.6625

Here, mode-3 has inversion symmetry while the other two modes lack that symmetry. Since these frequencies are in the infrared region, we do not normally perceive a color for water; it appears transparent. However, due to transitions made possible by the anharmonic effects, water molecules might possibly have some weak absorption in the visible range. As it is very weak, the absorption is noticeable only for large bodies of water, like the seas. There is a claim that the blue color of the seas is due to these weak absorption lines. Discuss if this explanation is plausible.

Visible range: $400 - 750\,\text{THz}$ $(0.4 - 0.7\,\mu\text{m})$

2. Electromagnetic Theory

2.1 EM-2018-2

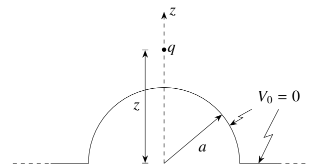
[EMT-2018-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) A conductor at potential $V_0 = 0$ has the shape of an infinite plane except for a hemispherical sphere of radius a. A charge q is placed above the center of a hemisphere at a distance from the center.

(i) Find the force on the charge q.

(ii) What would you do if you are asked to find the surface charge density on the flat part of the conductor? Give a brief description but do not compute anything.

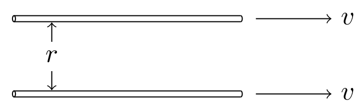


10 (b) Calculate the capacitance C of a spherical conductor of the inner radius a and the outer radius b which is filled with a dielectric material whose dielectric constant varying as

$$\varepsilon = \varepsilon_0 + \varepsilon_1 \cos^2\theta$$

where θ is the polar angle, ε_0 and ε_1 are constants.

10 (c) Consider two thin, parallel, very long, nonconducting rods, which are a distance r apart. They have identical constant charge density per unit length λ in their rest frames. If they move with velocity v (not necessarily small compared to the speed of light) with respect to the lab frame, calculate the force per unit length between them



(i) in the lab frame, and

(ii) in rods' rest frame.

(iii) Compare and interpret your results

10 (d) A static electric field $\vec{\boldsymbol{E}}$ is produced by some charges located in a region $\mathcal{R}$ as well as some external charges. Within the region $\mathcal{R}$, the charges have density $\rho(\vec{\boldsymbol{x}})$. The electric stress tensor is given by

$$T_{ij} = \epsilon_0 (E_i E_j - \frac{1}{2}\delta_{ij} E^2) \ .$$

Show that the total force acting on the charges inside the region $\mathcal{R}$ is given as

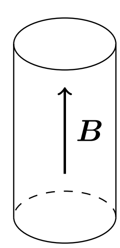
$$F_j = \int_{\mathcal{R}} \partial_i T_{ij} d^3\vec{\boldsymbol{x}} \ ,$$

where the integral is over the region $\mathcal{R}$.

Hint: What does the relation $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = 0$ imply for $\partial_i E_j$?

10 (e) An infinitely long solenoid with radius a produces uniform magnetic field inside

$$\boldsymbol{B} = \begin{cases} B\hat{\boldsymbol{z}} & \text{if } r < a \\ 0 & \text{if } r > a \end{cases} \ .$$



This field distribution has cylindrical symmetry.

(i) Find the vector potential $\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}})$ in the symmetric gauge, i.e.,

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{r}}) = f(r)\hat{\boldsymbol{\theta}} \ ,$$

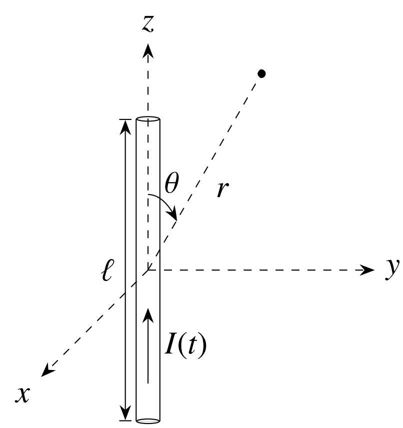
which also has cylindrical symmetry. Here $f(r)$ is a function of the cylindrical radial coordinate. Find the $f(r)$ inside $(r < a)$ and outside $(r < a)$ separately.

Hint: Express an appropriate closed line integral $\oint \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{\ell}}$ in terms of the magnetic flux.

(ii) If the magnetic field is increased with rate $\dot{B}$, find the induced electric field inside and outside.

[EMT-2018-Nov] Q2: Radiating Wire

Consider a thin, straight, conducting wire of length ℓ, centered on a given origin and oriented along the z axis as shown in the figure. The wire carries a oscillatory current $I(t) = I_0 \cos(\omega_0 t)$ everywhere along its symmetry axis. Let $\lambda_0 = 2\pi c/\omega_0$.



3 (a) Find the charges as a function of time and locate their positions.

5 (b) By choosing a suitable gauge, define the scalar and vector potentials.

15 (c) Obtain the scalar and vector potentials in the *far zone* (i.e, $\lambda_0 \ll \ell \ll r$ if r is radial distance from the origin O to the point of interest). Do not make any further simplifying assumption about the size of the wavelength λ_0.

3 (d) Find the magnetic field under the assumptions given in part (c).

5 (e) Find the power emitted per unit solid angle (or angular distribution of the radiation pattern).

4 (f) Calculate the electric dipole moment of the wire.

15 (g) Now concentrate on the long wavelength case ($\ell \ll \lambda_0 \ll r$, that is, we are still in the far radiation zone). Use part (f), calculate the power emitted per unit solid angle and compare with what you found in part (e).

2.2 EM-2018-1

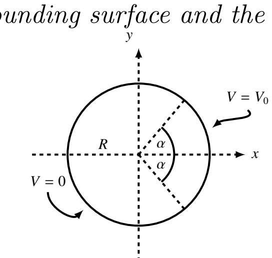
[EMT-2018-May] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

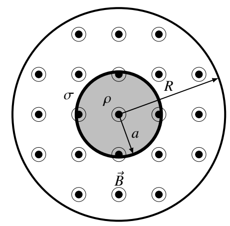
10 (a) Consider N point charges $q_1, q_2, \cdots, q_N$ located in a medium of dielectric constant ε_{in}, which is surrounded by another medium with dielectric constant ε_{out}. Determine the total polarization charge, Q_{pol} on the boundary between the two dielectrics.

Hint: Use Gauss law for the electric field $\vec{\boldsymbol{E}}$ inside and outside the bounding surface and the Gauss law for the electric displacement vector $\boldsymbol{D}$.

10 (b) Consider an infinite cylindrical conducting shell, whose cross-sectional view is given in the figure. Two thin insulating strips separate the cylinder into two parts. One segment is kept at the potential V_0, while the other segment is kept at zero potential. Determine the electrostatic potential inside the cylinder.



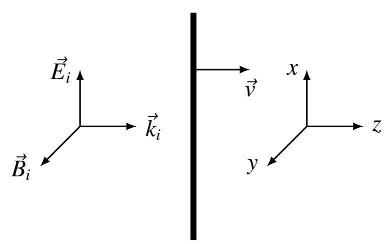
10 (c) A very long cylindrical solenoid of radius R and with its axis along the z-direction, maintains a magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{k}}$ in its interior (i.e. for $r \leq R$). A very long insulating cylinder with radius $a < R$ and permeability μ_0 is placed coaxially with the solenoid. The cylinder carries uniform volume charge density ρ and a uniform surface charge density $\sigma = -\frac{1}{2}a\rho$, making it neutral, as can be easily checked.



(i) Determine the electromagnetic angular momentum per unit length.

(ii) Suppose that the magnetic field of the solenoid is time dependent $\boldsymbol{B} = \boldsymbol{B}(t)$, and it is gradually switched off from its initial value B to zero. What is the induced electric field $\boldsymbol{E}_{ind}$ in terms of rate of change in $\boldsymbol{B}(t)$.

10 (d) A monochomatic plane wave with $\boldsymbol{E} = Ee^{ikz-i\omega t}\hat{\boldsymbol{x}}$ is incident on a plane mirror placed on the xy-plane as shown in the figure. The mirror moves at constant velocity $\boldsymbol{v} = v\hat{\boldsymbol{z}}$ in the $z-$ *direction* with respect to the lab frame. Show that the frequency ω_r and the wave vector $\boldsymbol{k}_r$ of the reflected wave in the lab frame are



$$\omega_r = \frac{1-\beta}{1+\beta}\omega\,, \quad \boldsymbol{k}_r = -\frac{1-\beta}{1+\beta}\boldsymbol{k}\,, \quad \beta = \frac{v}{c}\,.$$

Hint: One possible approach is to switch to the rest frame of the mirror. In this frame, what are ω'_r and $\boldsymbol{k}'_r$ of the reflected wave in terms of ω' and $\boldsymbol{k}'$ of the incident wave? Translate them back to the lab frame.

10 (e) Consider two particles with masses m_1 and m_2 and charges q_1 and q_2 moving with velocities much less than the speed of light.

(i) If the origin is chosen as the center of mass of the particles, show that the dipole moment of the system takes the form

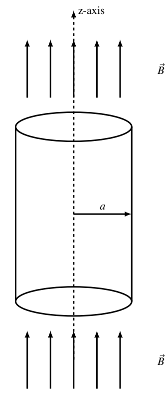
$$\boldsymbol{p} = \mu\left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right)\boldsymbol{r}\,,$$

where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ is the relative position vector and $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass.

(ii) The interaction between the particles is Coulombic. Determine the total instantaneous power radiated in the dipole approximation and express it as a function of the distance $r = |\boldsymbol{r}|$ between the particles.

Hint: Make use of the Newton's 2nd law for the relative motion in the last step of your calculation.

[EMT-2018-May] Q2: Taming the Magnetostatic Spins Waves



Consider a very long magnetizable cylinder of radius a with its symmetry axis extending along the z-direction. The cylinder is exposed to a uniform magnetic field $\boldsymbol{B} = B_z\hat{\boldsymbol{k}}$ in the z-direction. The induced magnetization of the cylinder has the form

$$\boldsymbol{M} = M_0\hat{\boldsymbol{k}} + \boldsymbol{m}e^{i\omega t}\,,$$

where $|\boldsymbol{m}| \ll M_0$ and the wavelength $\lambda = \frac{2\pi c}{\omega} \gg a$.
In words, the induced magnetization has small time dependent part leading to *magnetostatic spin waves*, since we can think of the magnetization of the cylinder due to sum of the spins and write $\boldsymbol{M} = \gamma\boldsymbol{S}$ where γ is the gyromagnetic factor.

Inside the the cylinder we have

$$\boldsymbol{B} = B_z\hat{\boldsymbol{k}} + \boldsymbol{b}e^{-i\omega t}\,, \quad \boldsymbol{H} = H_z\hat{\boldsymbol{k}} + \boldsymbol{h}e^{-i\omega t}\,, \quad \text{with} \quad \boldsymbol{H} = \boldsymbol{B} - 4\pi\boldsymbol{M}\,,$$

with $\boldsymbol{\nabla}\cdot\boldsymbol{b} = 0$ and $\boldsymbol{\nabla}\times\boldsymbol{h} = 0$.

In this problem our main goal is to determine the frequency ω of these waves in terms of the applied magnetic field B_z, M_0 and γ. To this end follow the steps below.

10 (a) Motion of the spins about the applied magnetic field leads to the precession equation for the magnetization, which is expressed as

$$\frac{d\boldsymbol{M}}{dt} = \gamma\boldsymbol{H}\times\boldsymbol{M}\,,$$

Using this equation in cylindrical polar coordinates show that (with the notation $\boldsymbol{m} = (m_r, m_\theta, m_z)$, $\boldsymbol{h} = (h_r, h_\theta, h_z)$)

$$m_r = \lambda_1 h_r - i\lambda_2 h_\theta\,, \quad m_\theta = i\lambda_2 h_r + \lambda_1 h_\theta\,, \quad m_z/m_r \sim \mathcal{O}(|\boldsymbol{m}|/M_0) \sim 0\,.$$

Determine the coefficients λ_1 and λ_2.

10 (b) Introduce the magnetic scalar potential ϕ by $\boldsymbol{h} = -\boldsymbol{\nabla}\phi$. Briefly explain why the equations satisfied by ϕ, inside and outside the cylinder take the form

$$\boldsymbol{\nabla}^2\phi_{out} = 0\,, \quad \boldsymbol{\nabla}^2\phi_{in} = 4\pi\boldsymbol{\nabla}\cdot\boldsymbol{m}\,,$$

where $\boldsymbol{\nabla}^2$ is the Laplacian in polar coordinates in two dimensions. Using the result of part (a) evaluate $\boldsymbol{\nabla}\cdot\boldsymbol{m}$ and simplify further the equation for ϕ_{in}.

10 (c) We may express the solutions for ϕ_{in} and ϕ_{out} as

$$\phi_{in} = \sum_{n=0}^{\infty}\frac{r^n}{a^n}(A_n e^{in\theta} + B_n e^{-in\theta})\,, \quad \phi_{out} = \sum_{n=0}^{\infty}\frac{a^n}{r^n}(C_n e^{in\theta} + D_n e^{-in\theta})\,.$$

Imposing the boundary conditions satisfied by h_θ and b_r, show that we must have

$$\sum_{n=0}^{\infty}\frac{n}{a}\left[(1+2\pi\lambda_1+2\pi\lambda_2)A_n e^{in\theta} + (1+2\pi\lambda_1-2\pi\lambda_2)B_n e^{-in\theta}\right] = 0\,.$$

10 (d) Using the result of the previous part determine the angular frequency ω of the spin waves.
Note: You do not need to determine (and should not try so!) the coefficients A_n and B_n.

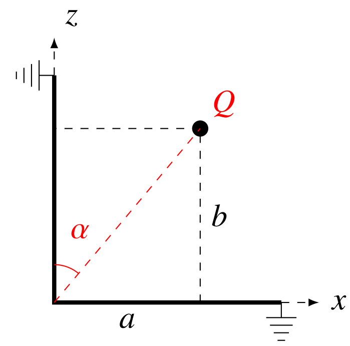
10 (e) For the n^{th} mode in the potential the tangential speed is $v = r\frac{\omega}{n}$, and it is indeed confirmed to be $\ll c$ in experiments. What is the electric polarization, say $\boldsymbol{p}'$, in the frame with instantaneous velocity $\boldsymbol{v}$ $(1/\sqrt{1-v^2/c^2} \sim 1)$.
Hint: Note that $\boldsymbol{p}$ and $\boldsymbol{m}$ may be considered to transform just like $\boldsymbol{E}$ and $\boldsymbol{B}$ under Lorentz transformations.

2.3 EM-2017-2

[EMT-2017-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) Consider two semi-infinite grounded conducting plates placed at right angles to each other as shown in the figure. We take the coordinate system such that the plates are on the xy- and yz- planes. A charge Q is placed at the point $(a, 0, b)$.

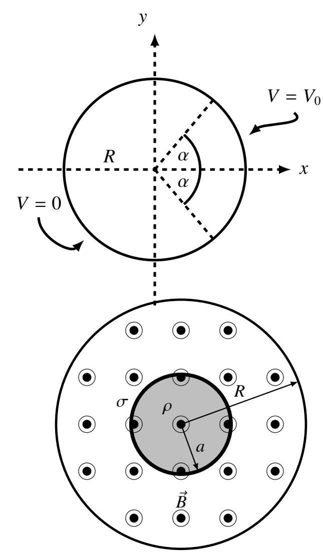


(i) Determine the image charge configuration and write down an expression for the electric potential at a point $P \equiv (x, y, z)$ in the region $x > 0$ and $z > 0$.

(ii) What are the induced charges on the plates extending along the positive x and z-axis. Explain your reasoning.

Hint: Think of the two extreme cases when the charge is very close to the xy-plane or the yz-plane.

10 (b) Consider a very long cylindrical solenoid of radius R with n turns per unit length and carrying a current I.



(i) What is the magnetic field $\boldsymbol{B}$ inside ($r < R$) and outside ($r > R$) the solenoid. Find the magnetic flux Φ everywhere.

(ii) Determine the vector potential $\boldsymbol{A}$ inside and outside solenoid.

(iii) Find a function χ such that $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi = 0$ for $r > R$. Considering the magnetic flux through a disk of radius $r > R$ perpendicular to the axis of the solenoid, what seems peculiar about the function χ.

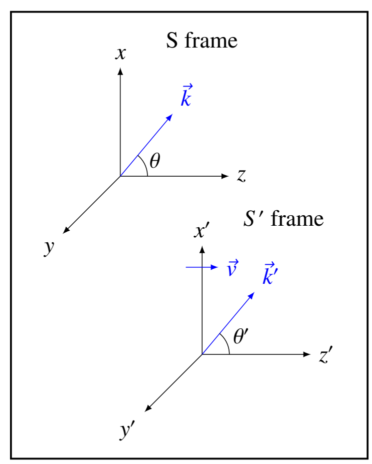
10 (c) Consider a ball of radius R with uniform polarization $\boldsymbol{P} = P\hat{\boldsymbol{k}}$.

(i) What are the volume and surface polarization charge densities $\rho_P = -\boldsymbol{\nabla} \cdot \boldsymbol{P}$ and $\sigma_P = \boldsymbol{P} \cdot \hat{\boldsymbol{r}}$, respectively.

(ii) Determine the total electrostatic energy stored in the system.

Hint: Determine the potential on the surface of the ball by a method of your choice and then use the formula for the electrostatic energy.

10 (d) We can determine the position of a star by the knowledge of the direction of the propagation vector of the light it emits. Consider two inertial observers S and S' where the S' observer is moving with respect to S with the constant velocity $\boldsymbol{v} = v\hat{\boldsymbol{z}}$. If the wave vectors of the light from a star are denoted by $\boldsymbol{k}$ and $\boldsymbol{k}'$ in S and S' frame, respectively and



$$\boldsymbol{k} \cdot \boldsymbol{v} = kv\cos\theta \qquad \boldsymbol{k}' \cdot \boldsymbol{v} = k'v\cos\theta' \,.$$

Show that

$$\cot\theta = \gamma\left(\cot\theta' + \frac{\beta}{\sin\theta'}\right), \text{ where } \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \,.$$

Hint: Explicitly write down the (inverse) Lorentz transformation from the K' to the K frame for the vectors $k^\mu = (k^0, \boldsymbol{k}^\perp, k^3)$ and $k'^\mu = (k'^0, \boldsymbol{k}'^\perp, k'^3)$, and use the elementary properties of electromagnetic waves.

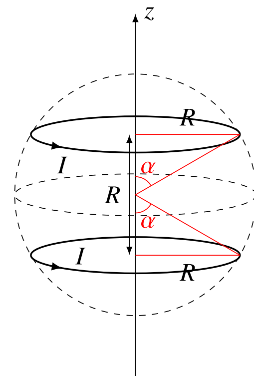
10 (e) Consider a particle performing circular motion with velocity $\boldsymbol{v}$ on the plane perpendicular to the magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{k}}$. Show that the total power radiated by the charge can be expressed as

$$P = \frac{2q^2}{3m^2c^5}\gamma^2 v^2 B^2 \,.$$

Hint: Recall that the total power radiated by an accelerating charge q is $P = -\frac{2}{3}\frac{q^2}{m^2c^3}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}$ and use the covariant form of the Lorentz Force law $\frac{dp^\mu}{d\tau} = \frac{q}{mc}F^{\mu\nu}p_\nu$. Here $d\tau = \frac{dt}{\gamma}$ is the proper time element.

[EMT-2017-Nov] Q2: Helmholtz Coils

Consider the set up of Helmholtz coils in which we have two parallel, coaxial and circular wire loops of radius R and separated by a distance R. Each loop carries a current I in the same direction as shown in the figure. Imagine a spherical surface on which the coils (wire loops) lie as shown by the dashed lines in the figure. From the simple geometry it is easy to obtain the radius a of the sphere as well as the values of $\sin\alpha$ and $\cos\alpha$. Use spherical coordinates as (r, θ, ϕ).



4 (a) The surface current density $\boldsymbol{K}$ can be expressed as

$$\boldsymbol{K} = \Lambda \sin\alpha \, \left(\delta(\cos\theta - \cos\alpha) + \delta(\cos\theta + \cos\alpha)\right) \hat{\boldsymbol{\phi}}\,,$$

where Λ is a constant. Show that $\Lambda = I/a$ by computing the total current of the coils.

18 (b) Compute the magnetic scalar potential, ψ, inside and outside the spherical surface as an infinite series. In particular show that asymptotically (i.e., as $r \to \infty$) the first two non-vanishing terms in the series expansion of ψ_{out} has the form

$$\psi_{out} = \frac{\lambda_1(\theta)}{r^2} + \frac{\lambda_2(\theta)}{r^6}\,.$$

and determine the functions $\lambda_1(\theta)$ and $\lambda_2(\theta)$.

Hint1: In order to solve this part

- *Observe the symmetry in the problem and write down the general forms for ψ_{in} and ψ_{out} as series with undetermined coefficients.*
- *Write down the boundary conditions that has to be satisfied by the components of the magnetic field $\boldsymbol{B}$ in terms of $\boldsymbol{K}$ and derivatives of ψ_{in} and ψ_{out} and solve for the unknown coefficients.*

6 (c) Compute the asymptotic form of the magnetic field $\boldsymbol{B}$ using the result of part b.

6 (d) Suppose that you want to place a second set of Helmholtz coils coaxially with the first to cancel the dipole part of the magnetic field found in part c. Using the principle of superposition determine the radius of the second pair of coils if they carry a current of $\frac{I}{4}$ each. What must be the direction of this current?

Now let us continue with the original set up of the problem (i.e. no second set of coils) and consider that the current in the coils has dependence on time in the form $I(t) = I_0 \sin(\omega t)$.

10 (e) Determine the instantaneous power radiated per unit solid angle $\frac{dP_m}{d\Omega}$ by the Helmholtz coils in the magnetic dipole approximation.

Hint2: Instantaneous power radiated per unit solid angle due to a magnetic dipole is given by

$$\frac{dP_m}{d\Omega} = \frac{\mu_0}{16\pi^2c^3}\left|\left(\frac{d^2\boldsymbol{m}}{dt^2}\times\hat{\boldsymbol{n}}\right)\times\hat{\boldsymbol{n}}\right|^2,$$

where $\hat{\boldsymbol{n}}$ is the unit vector from the origin along the direction at which the instantaneous power is calculated. In the dipole approximation we consider that the coils are essentially centered at the origin.

6 (f) What is the total power radiated by the coils in the magnetic dipole approximation?

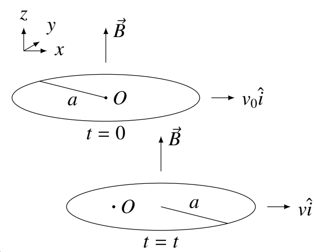
Hint3: $\int d\Omega \sin^2\theta = \frac{8\pi}{3}$.

2.4 EM-2017-1

[EMT-2017-May] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) A metal ring with radius a, mass M, and total resistance R is oriented to lie on the $x-y$ plane. The ring moves along the x-direction and its center passes through the origin with velocity $\boldsymbol{v} = v_0\hat{\boldsymbol{\imath}}$ at $t = 0$. The ring is immersed in a region of space with a magnetic field $\boldsymbol{B} = B_0 \frac{x}{x_0} \hat{\boldsymbol{k}}$. Assuming that $x_0 \ll a$, determine the distance the ring travels from the origin before it comes to rest.

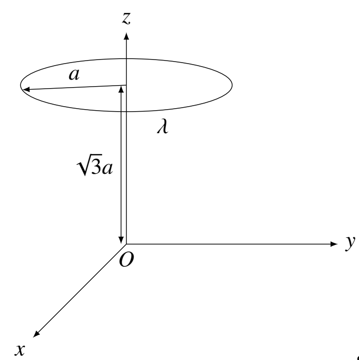


10 (b) Consider a charged ring of radius a with uniform charge density λ centered about the z-axis at $z = \sqrt{3}a$ as shown in the figure.

(i) What is the total charge Q on the ring.

(ii) Write down the volume charge density on the ring in terms of the total charge Q and suitable Dirac–δ functions in spherical coordinates with respect to the origin O as shown in the figure. Confirm that this correctly yields the total charge found in the previous part.

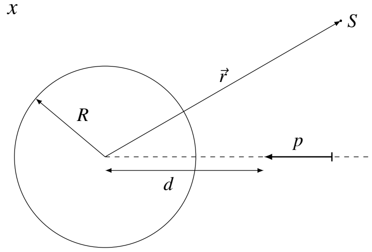
(iii) Evaluate the nonzero components of the dipole q_{1m}, and the quadrupole q_{2m} moments explicitly.



10 (c) An electric dipole $\boldsymbol{p}$ is placed at a distance $d\,(d > R)$ pointing toward the center of a conducting sphere of radius R. Consider that the dipole has length h and $|\boldsymbol{p}| = qh$. Using the method of images

(i) Determine the electric potential outside the sphere at a point S with $|\boldsymbol{r}| \gg h$, when the sphere is grounded.

(ii) Determine the electric potential at the point S, when the sphere is electrically isolated and neutral.



10 (d) An electron of charge e is released from rest and falls freely under the influence of gravity.

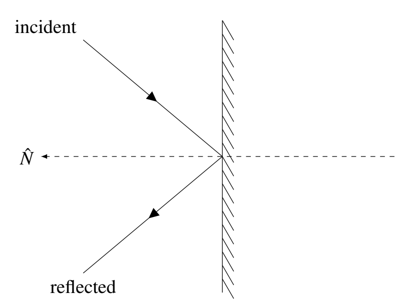
(i) Using Larmor formula determine the total energy radiated away by the charge after it travels a distance of 10 meters.

(ii) Determine the fraction of the potential energy lost in the form of radiation in the course of this motion. Do you think that we can safely neglect the energy loss due to radiation in this case? Explain why.

10 (e) A plane electromagnetic wave with wave vector $\boldsymbol{k} = k\hat{\boldsymbol{n}}$ is incident on a wall with incidence angle θ as shown in the figure. The wave is reflected with a reflection coefficient R. The energy momentum tensor for the incident wave can be written as

$$T^{\mu\nu} = \frac{u}{c^2\omega^2} k^\mu k^\nu e^{-2i\,k\cdot x}$$

where u stands for the energy density, ω for the frequency of the plane wave, and k^μ and x^μ are four vectors. The normal vector of the wall is $\hat{\boldsymbol{N}}$ as depicted on the figure.



(i) Write down the relation between the frequency and wave number of the plane wave.

(ii) What is the force per unit area $\boldsymbol{F}$ exerted on the wall in this process. Determine the component of this force normal to the wall. This is called the light pressure.

Hint: Note that $\boldsymbol{F}$ *must be related to the Maxwell Stress tensor part of* $T^{\mu\nu}$ *and use the principle of superposition.*

[EMT-2017-May] Q2: Average Hyperfine Interaction

In the hydrogen atom, there is an interaction between the intrinsic magnetic moments of the proton and the electron, which is called as the *hyperfine interaction.* The interaction energy can be described as

$$H_{\rm hf} = -\vec{\boldsymbol{\mu}}_{\boldsymbol{e}} \cdot \vec{\boldsymbol{B}}_p(\vec{\boldsymbol{x}})$$

where $\vec{\boldsymbol{\mu}}_e$ is the electron's magnetic moment, $\vec{\boldsymbol{x}}$ is the position of the electron and $\vec{\boldsymbol{B}}_p(\vec{\boldsymbol{x}})$ is the magnetic field due to the magnetic moment of the proton, $\vec{\boldsymbol{\mu}}_p$, at the position of the electron. For an electron in the 1s state, the average hyperfine energy can be expressed as

$$\langle H_{\rm hf}\rangle = -\vec{\boldsymbol{\mu}}_e \cdot \langle \vec{\boldsymbol{B}}_p\rangle \ ,$$

where $\langle \vec{\boldsymbol{B}}_p\rangle$ is the average magnetic field seen by the electron,

$$\langle \vec{\boldsymbol{B}}_p\rangle = \int \vec{\boldsymbol{B}}_p(\vec{\boldsymbol{x}})|\psi(r)|^2 \, d^3x$$

and $\psi(r)$ is the 1s wavefunction, which depends only on the radial coordinate r. Below, you will compute $\langle \vec{\boldsymbol{B}}_p\rangle$ and show that it depends only on the central value of the wavefunction, $|\psi(0)|^2$.

For simplicity, consider the proton's magnetic moment $\vec{\boldsymbol{\mu}}_p$ to be due to a current distribution $\vec{\boldsymbol{J}}$ inside the proton. At the end, we will treat the proton as a point particle.

20 (a) Consider the sphere with radius R which is centered at the proton. Compute the integral of $\vec{\boldsymbol{B}}_p(\vec{\boldsymbol{x}})$ on this sphere

$$\vec{\boldsymbol{c}}_R = \int_{r<R} \vec{\boldsymbol{B}}_p(\vec{\boldsymbol{x}})\, d^3x = R^2 \oint \hat{\boldsymbol{n}} \times \boldsymbol{A}\, d\Omega$$

and show that

$$\vec{\boldsymbol{c}}_R = \frac{2\mu_0}{3}\vec{\boldsymbol{\mu}}_p \ .$$

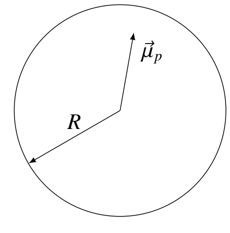
Observe that this is *independent* of R.

Hint: To perform this integral, carefully follow the steps given below:

1. *Insert a general integral expression for* $\boldsymbol{A}$ *in terms of the current density* $\boldsymbol{J}$ *and rearrange the integrals.*
2. *Perform the integral* $\int \frac{\hat{\boldsymbol{n}}}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d\Omega$ *using* $\hat{\boldsymbol{n}} = \sqrt{\frac{2\pi}{3}}(-Y_{11}^* + Y_{1-1}^*)\hat{\boldsymbol{\imath}} - \sqrt{\frac{2\pi}{3}}i(Y_{11}^* + Y_{1-1}^*)\hat{\boldsymbol{\jmath}} + \sqrt{\frac{4\pi}{3}}Y_{10}^*\hat{\boldsymbol{k}}$ *and the spherical harmonic expansion for* $\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}$.
3. *Use the definition for magnetic moment* $\vec{\boldsymbol{\mu}}_p$ *as an integral involving the current density to complete the calculation.*

8 (b) The magnetic field produced by the proton's dipole moment at a distant location $\vec{\boldsymbol{x}}$ can be expressed as

$$\vec{\boldsymbol{B}}_p(\vec{\boldsymbol{x}}) = \frac{\mu_0}{4\pi}\frac{3\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\mu}}_p) - \vec{\boldsymbol{\mu}}_p}{|\boldsymbol{x}|^3} \ . \qquad (1)$$



Show that at a fixed radius $R \neq 0$

$$\oint_{r=R} \vec{\boldsymbol{B}}_p(\vec{\boldsymbol{x}})\, d\Omega = 0 \, .$$

Hint: Consider making use of the result $\oint d\Omega\, \hat{n}_i \hat{n}_j = N\, \delta_{ij}$, *where* N *is a constant that you need to determine.*

Remark: Note that this result indicates that the average value of $\vec{\boldsymbol{B}}_p(\vec{\boldsymbol{x}})$ *due to* $\vec{\boldsymbol{\mu}}_p$ *is vanishing on a sphere of radius* R. *See the figure.*

8 (c) Results in parts (a) and (b) indicate that the magnetic field produced by the proton must have a Dirac-δ term that has to be added to Eq. (1). Make this correction to Eq. (1).

14 (d) Using the result of part (c), evaluate $\langle \vec{\boldsymbol{B}}_p \rangle$ and write down the value of the average hyperfine energy $\langle H_{\text{hf}} \rangle$ in terms of $\vec{\boldsymbol{\mu}}_e$, $\vec{\boldsymbol{\mu}}_p$ and $|\psi(0)|^2$.

2.5 EM-2016-2

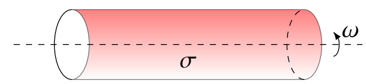
[EM-2016-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) A very long cylinder of radius R carries a uniform surface charge density σ and is set to rotate about its symmetry axis with angular speed ω.

(i) Determine the electric field inside and outside the cylinder.

(ii) Determine the magnetic field inside and outside the cylinder.

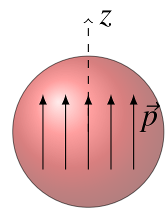


10 (b) Consider a sphere of radius R and with uniform polarization $\boldsymbol{P} = P\hat{\boldsymbol{k}}$.

(i) Determine the bound surface charge density $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{r}}$.

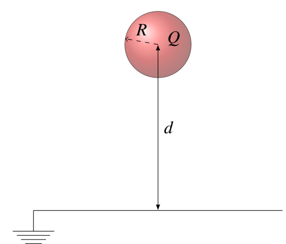
(ii) Determine the electric potential inside and outside the sphere.

(iii) What is the electric field inside the sphere.

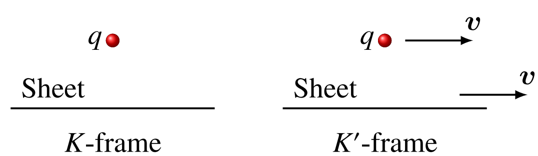


10 (c) A conducting sphere of radius R carrying a total charge Q is located at a distance d from a very large grounded conducting sheet on the xy plane. If $R \ll d$ then electric dipole approximation can be used where only the monopole and dipole terms are kept for the sphere. Using the method of images determine the electric potential at a point P above the xy plane with the dipole approximation.

Hint: Note that the problem has azimuthal symmetry about the z axis. What does this imply for the dipole moment $\mathbf{p}$?



10 (d) In a reference frame K, a stationary point charge q is located above a stationary infinite sheet of uniform surface charge density σ. With respect to an observer in another inertial reference frame K' both the point charge and the infinite sheet are moving with a relativistic velocity $\boldsymbol{v} = v\hat{\boldsymbol{y}}$ in the y direction.



(i) What is the electromagnetic force on the charge q in the K-frame?

(ii) Determine the electromagnetic fields created by the sheet in the K'-frame.

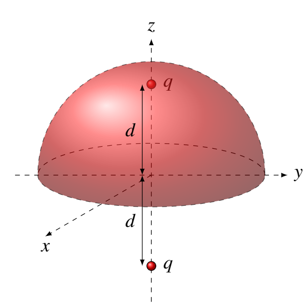
(iii) What is the total electromagnetic force on the point charge in the K'-frame?

10 (e) Consider two equal point charges placed along the z axis at $z = d$ and $z = -d$. Using the Maxwell stress tensor,

$$T_{ij} = \varepsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}|\boldsymbol{E}|^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij}|\boldsymbol{B}|^2 \right),$$

determine the electric force between the two charges. Does your answer agree with the result you get from the Coulomb's law?

Hint: Integrate the Maxwell stress tensor over a surface enclosing the charge at $z = d$. Simply consider that this surface is formed from the entire xy-plane and a hemisphere above it with very large radius and neglect the integral over the latter. For simplicity, use polar coordinates while integrating over the xy-plane.



[EM-2016-Nov] Q2: Dipole in Motion

Consider an ideal electric dipole with moment $\boldsymbol{p}$ moving with a non-relativistic velocity $\boldsymbol{v}(t)$. The dipole moment $\boldsymbol{p}$ has constant magnitude and a fixed direction in time. At a given time t its position is given by the vector $\boldsymbol{r}_0(t)$. Electric potential of this dipole may be expressed as

$$\Phi = -\boldsymbol{p} \cdot \boldsymbol{\nabla} \frac{1}{|\boldsymbol{x} - \boldsymbol{r}_0(t)|} ,$$

in Gaussian units.

8 (a) (i) Using the Poisson's equation for Φ, show that the charge density $\rho(\boldsymbol{x}, t)$ of the dipole may be expressed as

$$\rho(\boldsymbol{x}, t) = -\boldsymbol{p} \cdot \nabla \delta(\boldsymbol{x} - \boldsymbol{r}_0(t)) .$$

(ii) Show that the same charge density $\rho(\boldsymbol{x}, t)$ can be obtained starting from the polarization $\boldsymbol{P} = \boldsymbol{p}\, \delta(\boldsymbol{x} - \boldsymbol{r}_0(t))$ and then computing the corresponding bound charge density.

(iii) What is the corresponding current density $\boldsymbol{J}(\boldsymbol{x}, t)$ of the dipole in motion?

8 (b) Using the current density $\boldsymbol{J}(\boldsymbol{x}, t)$ determined in the previous part, show that the dipole in motion has a magnetic moment $\boldsymbol{m}$, which is given as

$$\boldsymbol{m} = \frac{1}{2c} \boldsymbol{p} \times \boldsymbol{v}(t) .$$

8 (c) Using the charge density determined in part (a), show that the dipole in motion has also an electric quadrupole moment Q_{ij}, which is given as

$$Q_{ij} = 3(r_{0i} p_j + r_{0j} p_i) - 2\boldsymbol{r}_0(t) \cdot \boldsymbol{p}\, \delta_{ij} .$$

Instantaneous power radiated per unit solid angle due to a magnetic dipole is given by

$$\frac{dP_m}{d\Omega} = \frac{1}{4\pi c^3} \left| \left(\frac{d^2 \boldsymbol{m}}{dt^2} \times \hat{\boldsymbol{n}} \right) \times \hat{\boldsymbol{n}} \right|^2 ,$$

where $\hat{\boldsymbol{n}}$ is the unit vector from the origin along the direction at which the instantaneous power is calculated. Whereas, the total power radiated by a *quadrupole* is given as $P_Q = \frac{1}{180c^5} \sum_{ij} \left| \frac{d^3 Q_{ij}}{dt^3} \right|^2$.

Let us now suppose that the dipole points in the z-direction, $\boldsymbol{p} = p\hat{\boldsymbol{z}}$, and rotates on a circle of radius R on the xy-plane with angular velocity ω. Thus we have $\boldsymbol{r}_0(t) = R \cos \omega t\, \hat{\boldsymbol{x}} + R \sin \omega t\, \hat{\boldsymbol{y}}$.

6 (d) Compute $\boldsymbol{m}$ and the non-vanishing components of Q_{ij} for the motion of the dipole specified above.

10 (e) Determine both the instantaneous power per unit solid angle, $\frac{dP_m}{d\Omega}$, and the total power P_m radiated due to the magnetic dipole moment found in part (d).

10 (f) Determine the total power P_Q radiated by the quadrupole moment found in part (d). Compare the frequency dependence of P_m and P_Q.

Hint 1: Recall that $\nabla^2 \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = -4\pi \delta(\boldsymbol{r} - \boldsymbol{x}')$.
Hint 2: In the calculation of $\boldsymbol{m}$ *and* Q_{ij}*, use index notation and integration by parts and note for the latter that the total derivative terms give no contribution.*
Hint 3: $\int d\Omega \sin^2 \theta = \frac{8\pi}{3}$.

3. Mathematical Methods in Physics

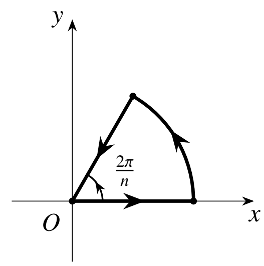
3.1 MP-2018-2

[MP-2018-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

9 (a) Which of the roots of the equation $z^n + 1 = 0$, where $n > 1$ is a positive integer, are enclosed by the circular sector shown in the figure? Use the residue theorem with the given contour in the figure to evaluate the real integral

$$\int_0^\infty \frac{dx}{1+x^n}.$$



13 (b) Chebyshev polynomials of 2nd kind satisfy the differential equation

$$(1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0.$$

(i) Put this equation into the self-adjoint form.
(ii) Specify the weighting factor and the interval of integration that guarantee that the Hermitian boundary conditions are satisfied.
(iii) Show that $U_n(x)$ are orthogonal for different n.

12 (c) Find the solution of the problem by using Fourier and Laplace transforms

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= \delta(x) \qquad (-\infty < x < \infty,\ t > 0) \\ u|_{t=0} &= 0, \\ \lim_{x\to\pm\infty} u &= 0. \end{aligned}$$

4 (d) The function $\psi(p) = \frac{d}{dp}\ln\Gamma(p)$ is called the digamma function. Show that

$$\psi(p+1) = \psi(p) + \frac{1}{p}.$$

12 (e) Consider the Poisson equation

$$\nabla^2 u(x,y) = f(x,y)$$

with homogeneous boundary conditions

$$u(0,y) = u(1,y) = 0,\ u_y(x,0) = u_y(x,1) = 0$$

in square region with unit side length in the xy–plane.

(i) Determine the eigenvalues and eigenfunctions of the Laplacian,

$$\nabla^2\varphi(x,y) = \lambda\,\varphi(x,y)\ ,$$

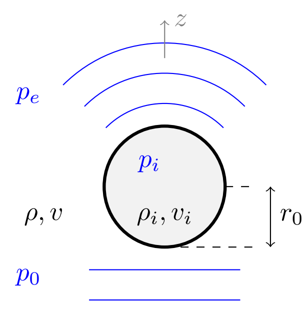
subject to the boundary conditions given above.

(ii) Construct the Green's function $G(x, y; x', y')$ in terms of the eigenfunctions found in part (i), and then write down the solution $u(x, y)$ of the Poisson equation.

[MP-2018-Nov] Q2: The scattering of a plane sound wave by a solid sphere...

If a sound wave encounters an obstacle in space it is partially reflected and partially transmitted through the obstacle. As a result of this, the initial direction of propagation of the wave is changed. This process is called scattering or diffraction of the sound waves. Consider a steady state sound wave that is set up in a homogeneous medium characterized by a density ρ and sound velocity v in which there is a homogeneous body (obstacle in the form of solid sphere) of density ρ_i and sound velocity v_i. The sound wave is characterized by the pressure p (the amplitude of the pressure oscillations) and the angular frequency ω of the acoustic vibrations. Assume that the medium occupies all space R with the exception of the volume Ω occupied by the body.

Let the field $p_0(\boldsymbol{r})$ be for the *incident* wave which would exist if the body were not there, the field $p_i(\boldsymbol{r})$ for the *refracted* wave within the body, and the field $p_e(\boldsymbol{r})$ for the *reflected* or *scattered* wave such that the actual so-called acoustic wave with pressure field $p(\boldsymbol{r})$ is the superposition of the *incident* and the *reflected* waves, $p(\boldsymbol{r}) = p_0(\boldsymbol{r}) + p_e(\boldsymbol{r})$.
Because of the homogeneity of the body and of the medium, the pressure at their internal points satisfies the Helmholtz equations. Assuming that the incident wave is given, one can get the following problem:



$$\nabla^2 p_i + k_i^2 p_i = 0, \quad \boldsymbol{r} \in \Omega \setminus \partial\Omega \tag{2}$$
$$\nabla^2 p_e + k^2 p_e = 0, \quad \boldsymbol{r} \in R \setminus \Omega \tag{3}$$

where $k_i^2 = \omega^2/v_i^2,\ \ k^2 = \omega^2/v^2$. The scattered wave $p_e(\boldsymbol{r})$ satisfies the *radiation* condition as $r \to \infty$:

$$\lim_{r\to\infty} r\left(\frac{\partial p_e}{\partial r} - ikp_e\right) = 0, \quad \lim_{r\to\infty} p_e = 0. \tag{4}$$

The pressure and velocity of vibrations in the body and in the medium coincide each other at the boundary $\partial\Omega$ of the body, which leads to the following conditions:

$$p_i = p_0 + p_e,\ \ \frac{1}{\rho_i}\frac{dp_i}{dn} = \frac{1}{\rho}\frac{dp_0}{dn} + \frac{1}{\rho}\frac{dp_e}{dn}, \quad \boldsymbol{r} \in \partial\Omega, \tag{5}$$

where d/dn denotes the normal derivative. Consider the problem (2)-(5) with the assumption that the region Ω is a sphere of radius r_0 and that the incident wave is a plane wave

$$p_0(r, \theta) = A\, e^{ikr\, cos\theta}. \tag{6}$$

Here (r, θ, ϕ) are the spherical polar coordinates and $p_0(r, \theta)$ has azimuthal symmetry (that is, independent of ϕ) if the center of the sphere Ω is chosen as the origin of the coordinate system. The solution of the diffraction problem will also possesses the symmetry of p_0 so that the regions indicated as $\Omega\setminus\partial\Omega$, $R\setminus\Omega$, and $\partial\Omega$ divide the space into three parts $r < r_0$, $r > r_0$, and $r = r_0$, respectively.

10 (a) Show that series expansion of the scattered wave takes the form

$$p_e = \sum_{m=0}^{\infty} a_m h_m^{(1)}(kr) P_m(cos\theta), \quad h_m^{(1)}(kr) = \sqrt{\pi/2kr}\, H^{(1)}_{m+\frac{1}{2}}(kr), \tag{7}$$

where $H^{(1)}_{m+\frac{1}{2}}(kr)$ is a Hankel function of the first kind.

10 (b) Show that series expansion of the refracted wave takes the form

$$p_i = \sum_{m=0}^{\infty} b_m j_m(k_i r) P_m(cos\theta), \quad j_m(k_i r) = \sqrt{\pi/2k_i r}\, J_{m+\frac{1}{2}}(k_i r), \tag{8}$$

where $J_{m+\frac{1}{2}}(k_i r)$ is a Bessel function.

10 (c) Use the boundary conditions (5) to obtain the equations below which can be used for finding the coefficients a_m and b_m ($m = 0, 1, 2, 3, ...$):

$$b_m j_m(k_i r_0) - a_m h_m^{(1)}(kr_0) = A(2m+1)\, i^m\, j_m(kr_0)\,, \tag{9}$$

$$\frac{k_i}{\rho_i} b_m j'_m(k_i r_0) - \frac{k}{\rho} a_m h_m^{(1)\prime}(kr_0) = A\frac{k}{\rho}(2m+1)\, i^m\, j'_m(kr_0)\,. \tag{10}$$

Here the superscript $'$ indicates derivatives with respect to argument of the functions.
Hint: These equations can be derived by using the expansion for a plane wave

$$e^{ikr\,cos\theta} = \sum_{m=0}^{\infty} (2m+1)\, i^m\, j_m(kr) P_m(cos\theta).$$

10 (d) Now concentrate on the problem of the scattering of a plane sound wave (6) from an absolutely rigid immovable sphere Ω of radius r_0, and obtain the expression

$$p_e = -A \sum_{m=0}^{\infty} (2m+1)\, i^m\, \frac{j'_m(kr_0)}{h_m^{(1)\prime}(kr_0)} h_m^{(1)}(kr) P_m(cos\theta)\,, \tag{11}$$

which can be obtained by setting $\rho_i \to \infty$ in (5) and (10). Show that under this condition the equation (5) transforms into the form

$$\frac{dp_e}{dr} = A\, ik\, cos\theta\, e^{ikrcos\theta} \quad (at\ r = r_0). \tag{12}$$

10 (e) Finally, show that when the wavelength is large, compared with the dimensions of the sphere Ω ($kr_0 \ll 1$), the solution of the preceding problem at a large distance from the sphere ($kr \gg 1$) can be represented by the form

$$p_e \xrightarrow{kr\gg 1} -\frac{Ak^2 r_0^2}{3r}\left(1 - \frac{3}{2}\cos\theta\right) e^{ikr}.$$

Use the asymptotic forms of Bessel and Hankel functions for small and large argument limits.
Hint: You may find useful relations in the formulae sheet.

3.2 MP-2018-1

[MP-2018-May] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) The contour integral definition of the spherical Hankel function of the 1st kind is

$$h_\ell^{(1)}(x) = \frac{(-1)^\ell 2^\ell \ell!}{\pi x^{\ell+1}} \oint_C \frac{e^{-ixz}dz}{(z^2-1)^{\ell+1}},$$

where C is a contour enclosing the point $z = -1$ only. Using residue calculus, show that $h_\ell^{(1)}(x)$ takes the form

$$h_\ell^{(1)}(x) = \sum_{k=0}^{\ell} A_k^\ell \frac{e^{ix}}{x^{k+1}},$$

and identify the constant coefficients A_k^ℓ.
Hint: You may find useful relations in the formulae sheet.

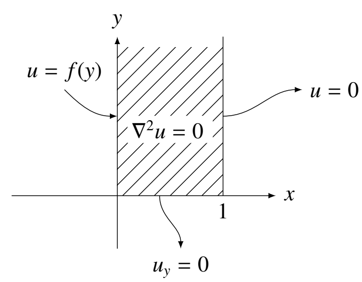
10 (b) Transform the eigenvalue problem for the Bessel equation

$$x^2y'' + xy' - 16y + \lambda x^2 y = 0, \quad 0 < x < 5$$
$$|y(0)| < \infty,\ y(5) = 0.$$

into the self-adjoint form. Find the eigenvalues and eigenfunctions. Determine the weight factor that makes the eigenfunctions orthogonal.

10 (c) Use the Fourier cosine transformation to solve a Dirichlet-Neumann problem for the Laplace equation in a semi-infinite string:

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < 1,\ y > 0,$$
$$u_y(x, 0) = 0, \qquad 0 < x < 1,$$
$$u(0, y) = f(y),\ u(1, y) = 0, \qquad y > 0.$$



8 (d) The complex factorial function $z!$ for $\mathrm{Re}(z) \le -1$ is defined by

$$z! = \frac{(z+n)!}{(z+n)(z+n-1)...(z+1)},$$

where n is any (positive) integer greater than $-\mathrm{Re}(z)$.

(i) Show that $z!$ (with $\mathrm{Re}(z) \le -1$) is the same for any value of n.

(ii) Prove that the residue of $z!$ at the pole $z = -m$, where m is a positive integer, is $(-1)^{m-1}/(m-1)!$.

12 (e) (i) Verify that the variational form of the problem

$$(p(x)y')' - q(x)y + \lambda\, r(x)y = 0,\ \ y(a) = y_a,\ y(b) = y_b,$$

is $\delta I[y] = \delta \int_a^b \left[p(x)y'^2 + q(x)y^2 - \lambda\, r(x)y^2\right] dx = 0$, or, equivalently,
$\delta J[y] = \delta \int_a^b \left[p(x)y'^2 + q(x)y^2\right] dx = 0$ subject to the constraint $N[y] = \int_a^b y^2\, r(x)dx = \mathrm{Const.}$

(ii) Let $K[y] = J[y]/N[y]$. Deduce that functional K satisfying $\delta K[y] = 0$ reduces to the eigenvalue λ.

(iii) Consider the eigenvalue problem

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \lambda\, y = 0\,, \quad y(-1) = y(1) = 0\,.$$

Derive an approximation to the lowest eigenvalue λ_0, by using the trial function $\varphi(x) = 1 - x^2$.

[MP-2018-May] Q2: The Born Approximation For Scattered Waves...

According to the quantum-mechanical theory of scattering, the wavefunction $\psi(\mathbf{r})$ of particles moving along the negative $z-$axis (toward the origin) and scattered by a potential, $V(\mathbf{r})$, must satisfy the time-independent Schrödinger equation $-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\,\psi(\mathbf{r})$. Let us look for a solution having an asymptotic form

$$\psi(\mathbf{r}) \sim e^{i\chi z} + g_\chi(\theta,\phi)\frac{e^{i\chi r}}{r} = e^{i\chi z} + \Phi(\mathbf{r}). \tag{13}$$

Then, the Born approximation solution can be obtained from the equation

$$\left(\nabla^2 + \chi^2\right)\Phi(\mathbf{r}) = f(\mathbf{r})\,, \tag{14}$$

where $\chi^2 = 2mE/\hbar^2$ and $f(\mathbf{r}) = \frac{2m}{\hbar^2}V(\mathbf{r})e^{i\chi z}$.

10 (a) Using equation (14), show that the Fourier transform $\hat{\Phi}(\mathbf{k})$ of $\Phi(\mathbf{r})$ is

$$\hat{\Phi}(\mathbf{k}) = -\frac{\hat{f}(\mathbf{k})}{k^2 - \chi^2},$$

where $\hat{f}(\mathbf{k})$ is the Fourier transform of $f(\mathbf{r})$.

10 (b) Apply the inverse Fourier transform to $\hat{\Phi}(\mathbf{r})$ to obtain

$$\Phi(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d^3r',$$

where the Green's function is

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2-\chi^2}d^3k.$$

10 (c) Taking the direction of the vector $(\mathbf{r}-\mathbf{r}')$ as the polar axis for a $\mathbf{k}-$space integration in spherical (k,θ,φ) coordinates, such that $\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') = k|\mathbf{r}-\mathbf{r}'|\cos\theta$, reduce $G(\mathbf{r},\mathbf{r}')$ to (recall that $d^3k = k^2\sin\theta d\theta d\varphi$)

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi^2 i|\mathbf{r}-\mathbf{r}'|}\int_0^\infty \frac{e^{ik|\mathbf{r}-\mathbf{r}'|} - e^{-ik|\mathbf{r}-\mathbf{r}'|}}{k^2-\chi^2}kdk.$$

As the integrand is an even function of k, with notations $w = k|\mathbf{r}-\mathbf{r}'|$ and $w_0 = \chi|\mathbf{r}-\mathbf{r}'|$, this integral can also be written in the form

$$-\frac{1}{4\pi^2|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{\infty}\frac{w\sin w}{w^2-w_0^2}dw. \tag{15}$$

10 (d) Evaluate the integral (15) from part (c) by a contour integration and show that it takes the form

$$G(\mathbf{r},\mathbf{r}') = -\frac{e^{i\chi|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}.$$

10 (e) Use this result and the asymptotic form given in equation (13) to determine the scattered wave-function, $\psi(\mathbf{r})$, in the Born approximation for a potential $V(\mathbf{r}) = V_0\,\delta(r)$.

3.3 MP-2017-2

[MP-2017-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) Show that, if a is a positive real constant, $f(z) = e^{iaz^2}$ is analytic and $f(z) \to 0$ as $|z| \to \infty$ for $0 < \arg(z) \leq \pi/4$. By applying Cauchy's theorem to a suitable contour, find

$$\int_0^\infty \cos(ax^2)dx.$$

10 (b) The ultraspherical (Gegenbauer) polynomials $C_n^{(\alpha)}(x)$ are solutions of the differential equation

$$\left[(1-x^2)\frac{d^2}{dx^2} - (2\alpha+1)x\frac{d}{dx} + n(n+2\alpha)\right] C_n^{(\alpha)}(x) = 0.$$

Transform this equation into self-adjoint form. Show that the $C_n^{(\alpha)}(x)$ are orthogonal for differnt n. Specify the interval of integration and the weighting factor.

10 (c) Use Fourier transormation method to solve the problem

$$\begin{aligned} \cos(t)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} &= 0, \ -\infty < x < \infty, \ t > 0 \\ u(x,0) = f(x)\,. \end{aligned}$$

10 (d) There are equations, known as integro-differential equations, in which both derivatives and integrals of the unknown function appear. Solve the given integro-differential equaton

$$\varphi'(x) + 4\int_0^x \varphi(x-t)e^{-4t}dt = 16, \quad \varphi(0) = 0.$$

Hint: Use the Laplace transformation.

10 (e) Find functions $y(t)$ and $z(t)$ such that

$$\delta\left(\int_0^{\pi/4} \left(4\,y\,z - y_t^2 - 4z_t^2\right)dt\right) = 0,$$

subject to the conditions $y(0) = 0$, $y(\pi/4) = 0$ and $z(0) = 0$, $z(\pi/4) = 1$. Here $y_t = \frac{dy}{dt}$ and $z_t = \frac{dz}{dt}$.

[MP-2017-Nov] Q2: Fractional Modified Bessel Functions

Modified Bessel functions are solutions of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - \left(x^2 + n^2\right) y = 0,$$

and the generating function for the modified Bessel functions of the first kind, $I_n(x)$, is

$$e^{\frac{x}{2}\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} I_n(x) t^n.$$

5 (a) Verify the identity

$$\cosh x = I_0(x) + 2\sum_{n=1}^{\infty} I_{2n}(x).$$

10 (b) Show that

$$I_n(x) = \frac{1}{2\pi i} \oint_C e^{(x/2)(t+1/t)} \frac{dt}{t^{n+1}},$$

where n is an integer and C is any closed contour encircling the singularity at $t = 0$. If we choose the contour C to be a unit circle centered at the origin (i.e., substituting $t = e^{i\theta}$), $I_n(x)$ can be expressed as a real integral. Then, show that it can be written in the form

$$I_n(x) = \frac{1}{\pi} \int_0^{\pi} e^{x\cos\theta} \cos(n\theta) d\theta.$$

15 (c) The integral representation for the modified Bessel function $I_\nu(x)$ for $x > 0$ and non-integer ν can be given as

$$I_\nu(x) = \frac{1}{2\pi i} \int_C e^{(x/2)(t+1/t)} \frac{dt}{t^{\nu+1}},$$

where C is the contour shown in the Figure 1. Verify that $I_\nu(x)$ defined in this form satsfies the modified Bessel's equation.

10 (d) Substituting $u = \frac{1}{2} e^{i\pi} x t$, show that function $I_\nu(x)$ can be approximated (for $|x/t| \ll 1$) as

$$I_\nu(x) \approx \frac{1}{2\pi i} \left(\frac{x}{2}\right)^{\nu} e^{i\nu\pi} \int_{C'} \frac{e^{-u}}{u^{\nu+1}} du.$$

In Figure 2, **draw** the contour C' **on the complex** u**-plane**.

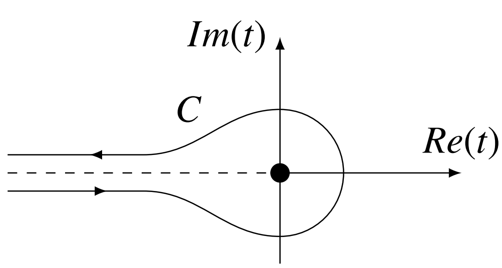


Figure 1: Contour C for the Bessel function I_ν.

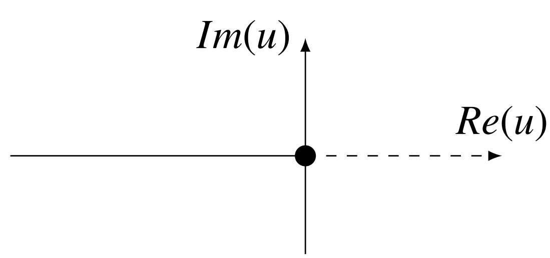


Figure 2: Contour C' for the Bessel function I_ν.

10 (e) Show that on the contour C'

$$\int_{C'} e^{-u} u^{\nu} du = 2ie^{i\nu\pi}\Gamma(\nu+1)\sin(\nu\pi),$$

and hence

$$I_\nu(x) \approx \left(\frac{x}{2}\right)^{\nu} \frac{\sin\left[(\nu+1)\pi\right]\ \Gamma(-\nu)}{\pi},$$

and then by the reflection formula for the gamma function

$$I_\nu(x) \approx \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu},$$

which is limiting form for small argument.

3.4 MP-2017-1

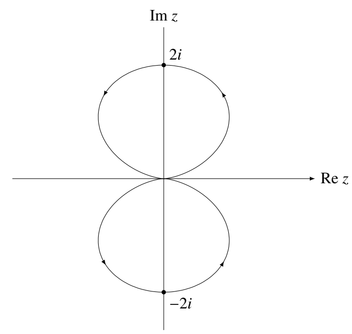
[MP-2017-May] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) Evaluate the complex integral

$$\oint_C \frac{dz}{z^4 - 1}$$

over the contour C shown in the figure.



10 (b) Consider two point charges of unit strength ($q = 1$) located on the z-axis at $z = 1$ and $z = -1$.

(i) Write down the electric potential $\Phi(\boldsymbol{x})$ at any point on a unit sphere (except $z = \pm 1$) and expand it in a series of Legendre polynomials in the form $\Phi(\boldsymbol{x}) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cos\theta)$. Determine the coefficients c_ℓ.

(ii) Evaluate the series $\sum_{j=0}^{\infty} P_{2j}(0)$.

Hint: Recall that we have the formula

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)\,,$$

where $cos\gamma = \hat{x} \cdot \hat{x}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$.

10 (c) Evaluate the integral

$$J_m = \int_0^\infty \frac{x^{m-1}}{e^x - 1} dx\,, \quad m > 0$$

by expanding an appropriate geometric series and using the definition of $\Gamma(m)$ and $\zeta(m)$.

Hint: The Riemann zeta function is $\zeta(m) = \sum_{n=1}^{\infty} n^{-m}$.

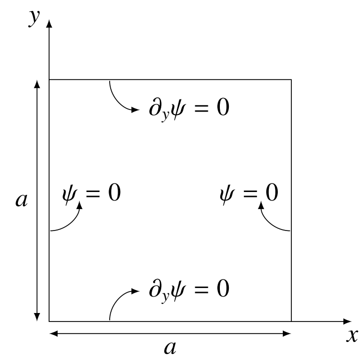
10 (d) In the quantum mechanics exam one of the problems dealt with the Yukawa potential of the form $V(\boldsymbol{r}) = -\frac{e^2}{r} e^{-\mu r}$ where $\mu > 0$ is a constant. In quantum mechanics it becomes useful to know the Fourier transform of $V(r)$ to compute the so called Schrödinger's integral equation. Evaluate the three dimensional Fourier transform

$$V(\boldsymbol{q}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int V(\boldsymbol{r}) e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} d^3r\,.$$

10 (e) Consider the differential equation

$$\nabla^2\psi + \lambda^2\psi = 0\,, \qquad \lambda > 0\,,$$

subject to the homogenous boundary conditions $\psi(0,y) = \psi(a,y) = 0$ and $\partial_y\psi(x,0) = \partial_y\psi(x,a) = 0$ in a square region of side length a, as given in the figure.



(i) Determine the characteristic values and characteristic functions.

(ii) Construct the corresponding Green's function $G(x, x\prime, y, y')$ in terms of the characteristic functions found in part (i).
Hint: Green's function $G(x, x', y, y')$ satisfies $\boldsymbol{\nabla}^2 G = -\delta(x - x')\,\delta(y - y')$.

[MP-2017-May] Q2: Legendre Functions of the Second Kind

Consider the Legendre's differential equation

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0\,, \quad \ell = 0, 1, 2, \cdots$$

Legendre polynomials $P_\ell(x)$ which may be given in terms of the Rodrigues representation as

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2 - 1)^\ell\,, \quad x \in [-1\,,1]\,.$$

are solutions to this differential equation in the interval $[-1, 1]$. In this problem we are going to explore solutions to this differential equation in the same interval, which are linearly independent from $P_\ell(x)$ and thus named Legendre functions of the second kind.

12 (a) Consider Legendre's differential equation for $\ell = 0$. By elementary methods show that the solution is

$$\frac{1}{2}\ln\frac{1 + x}{1 - x}\,.$$

Note: No credit will be given if you simply verify it by substituting this expression into the differential equation.

(i) Using complex function techniques show that

$$\frac{1}{2}\ln\frac{1 + x}{1 - x} = \tanh^{-1} x\,.$$

10 (b) Consider the functions $Q_\ell(x) = P_\ell(x)\tanh^{-1} x + \mathit{\Pi}_\ell(x)$. Obtain the inhomogeneous differential equation satisfied by $\mathit{\Pi}_\ell(x)$ and argue from your result that $\mathit{\Pi}_\ell(x)$ must be a polynomial in x. What is the degree of this polynomial.

8 (c) As a consequence of part b., $\mathit{\Pi}_\ell(x)$ can be expanded in terms of the Legendre polynomials in the interval $[-1, 1]$ as

$$\mathit{\Pi}_\ell(x) = \sum_{n=0}^{\ell-1} c_n P_n(x)\,.$$

Using the differential equation obtained in part (b), evaluate $\mathit{\Pi}_2(x)$.

10 (d) An integral expression for $Q_\ell(x)$ has the form

$$Q_\ell(x) = \frac{1}{2^{\ell+1}}\int_{-1}^{1} dt \frac{(1 - t^2)^\ell}{(x - t)^{\ell+1}}$$

Integrating this expression by parts ℓ times show that another integral representation of $Q_\ell(x)$ is

$$Q_\ell(x) = \frac{1}{2}\int_{-1}^{1} dt \frac{P_\ell(t)}{x-t}$$

10 (e) Using the integral form obtained in part d. show that

$$Q_\ell(x) = P_\ell(x)\,\tanh^{-1} x + \Pi_\ell(x)$$

Hint: Consider adding and subtracting an appropriate term to the numerator in the integrand of $Q_\ell(x)$*. Do not attempt to calculate* $\Pi_\ell(x)$*, just leave it as an integral.*

3.5 MP-2016-2

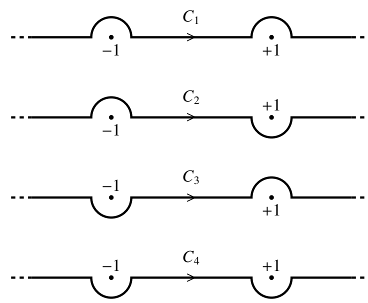
[MP-2016-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) Evaluate the complex integral

$$\int_{C_i} \frac{dz}{z^2-1}$$

over the contours C_i, $i = 1, 2, 3, 4$, shown in the diagram. In each case, first close the contour in a way you choose and then use residue methods.



10 (b) Consider the complex functions

$$f(z) = \sinh\left(\frac{4}{z}\right), \qquad g(z) = \frac{1}{(z-i)(z-2)}.$$

(i) Determine the Laurent series expansion for these functions around the origin of the complex plane.

(ii) What is the residue of $f(z)$ at its singularity.

10 (c) In a certain quantum mechanics problem the wave function at the origin takes the form

$$\psi(0) = e^{\frac{-\pi a}{2}}\,\Gamma(1+ia)\,.$$

Using the properties of the gamma function, show that

$$\psi^*(0)\psi(0) = \frac{2\pi a}{e^{2\pi a}-1}\,.$$

Hint: Recall that $\Gamma(z+1) = z\Gamma(z)$ *and* $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}$.

10 (d) Following Green's function appears in the discussion of forced, damped harmonic oscillator

$$G(t) = \begin{cases} \frac{1}{\Omega} e^{-\gamma t}\sin(\Omega t) & t > 0 \\ 0 & t < 0 \end{cases}.$$

where Ω and γ are positive constants. Find the Fourier transform $g(\omega)$ of $G(t)$. Where are the poles of $g(\omega)$ located on the complex plane with $\mathcal{R}e(z) = \omega$.

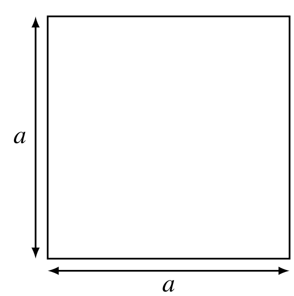
10 (e) Consider the differential equation

$$\nabla^2\psi + \lambda^2\psi = 0\,,$$

subject to the homogenous boundary conditions $\partial_x\psi(0,y) = \partial_x\psi(a,y) = \partial_y\psi(x,0) = \partial_y\psi(x,a) = 0$ in a square region of side length a, as in Figure 1 given below.

(i) Determine the characteristic values and characteristic functions.

(ii) Construct the corresponding Green's function $G(x, x\prime, y, y')$ in terms of the characteristic functions found in part (i).
Hint: Green's function $G(x, x', y, y')$ *satisfies* $\boldsymbol{\nabla}^2 G = -\delta(x-x')\,\delta(y-y')$.



[MP-2016-Nov] Q2: Legendre Polynomials All the Way

Legendre polynomials may be defined using the Rodrigues representation as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\,, \quad x \in [-1\,,1]\,.$$

8 (a) Using the Rodrigues representation, compute $P_n(1)$ and $P_n(-1)$.

10 (b) Using the Rodrigues representation and integration by parts prove the orthogonality of Legendre polynomials, that is, show that

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm}\,.$$

8 (c) Using the results of part a) and b), show that

$$\delta(1-x^2) = \frac{1}{2}\sum_{n=0}^{\infty}(4n+1)P_{2n}(x)\,.$$

6 (d) Deduce from the Rodrigues formula the contour integral representation for the Legendre polynomials as

$$P_n(z) = \frac{1}{2^n}\frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+1}}dt\,,$$

where C encloses the point z.

8 (e) For the contour C in the previous part take a circle of radius $|\sqrt{z^2-1}|$ centered around the point z and show that a real integral representation of Legendre polynomials may be written as

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi}(z+\sqrt{z^2-1}\cos\alpha)^n d\alpha\,,$$

10 (f) Recall that the spherical harmonics $Y_{lm}(\theta,\phi)$ and the associated Legendre polynomials $P_l^m(x)$ may be expressed as

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}\,, \quad P_l^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l(x)\,.$$

Consider the angular momentum operator $\boldsymbol{L} = -i\boldsymbol{x}\times\boldsymbol{\nabla}$ and recall that $L_+ = L_1 + iL_2$. Compute

$$L_+ Y_{l0}(\theta,\phi)\,,$$

and express your answer in terms of the spherical harmonics. Note that this is a familiar result in quantum mechanics.

Hint: Do NOT attempt to express L_+ in spherical coordinates.

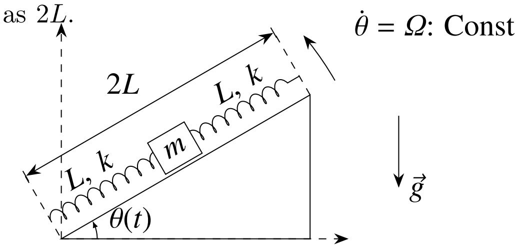
4. Analytical Mechanics

4.1 AN-2018-2

[AN-2018-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

14 (a) A block of mass m is free to slide along a inclined plane. The block is attached to the bottom and top points of the plane by two identical massless springs with stiffness k and relaxed length L as shown in the figure. The length of the inclined plane is given as $2L$. The plane itself is set to a uniform rotation with angular velocity Ω by an external agent and the block remains in contact with the surface of the plane at all times during the course of motion. The motion takes place under an uniform gravitational field $\boldsymbol{g}$ pointing downward. There is no friction in the system. Use *the D'Alembert's principle only*, obtain the equations of motion.



12 (b) For two particles moving in one-dimensional space, let x_i, p_i and m_i $(i = 1, 2)$ be the positions, momenta and masses respectively. We define a canonical transformation to the *relative* and *center-of-mass* coordinates and momenta as follows:

$$x_{\rm rel} = x_1 - x_2 \qquad p_{\rm rel} = A_1 p_1 + A_2 p_2 \ ,$$
$$x_{\rm cm} = (m_1 x_1 + m_2 x_2)/(m_1 + m_2) \qquad p_{\rm cm} = B_1 p_1 + B_2 p_2 \ ,$$

where A_i and B_i $(i = 1, 2)$ are some constants.

(i) Using Poisson brackets, find the constant coefficients so that the transformation $(x_1, p_1, x_2, p_2) \longrightarrow (x_{\rm rel}, p_{\rm rel}, x_{\rm cm}, p_{\rm cm})$ is canonical.

(ii) Find a $F_2(x_1, x_2, p_{\rm rel}, p_{\rm cm})$ type generating function.

(iii) The Hamiltonian of these particles is

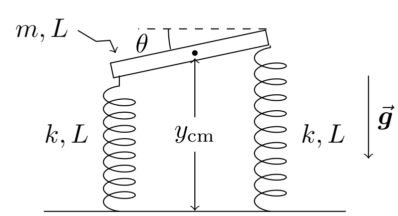
$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x_2 - x_1) \ .$$

Show that, H can be expressed as

$$H = \frac{p_{\rm rel}^2}{2m_{\rm rel}} + \frac{p_{\rm cm}^2}{2m_{\rm cm}} + V(x) \ ,$$

and find the corresponding masses.

10 (c) A uniform thin rod of mass m and length L is connected to the floor by two identical massless vertical springs of constant k and relaxed length L as shown in the figure. The rod is pulled from one end by a small amount and then released. Use Euler's equation for rigid body motion to obtain the equation of motion.



14 (d) A particle of mass m moves in a plane (take the xy-plane with y being vertically up direction) along a curve parametrized as

$$x(\theta) = L\,(2\theta + \sin 2\theta) \ , \qquad y(\theta) = L\,(1 - \cos 2\theta)$$

where L is a parameter in length dimension. Here the angle θ can be as large as $\pi/2$ which is the particle turning point. The motion takes place under uniform gravitational field $\boldsymbol{g}$ pointing in vertically downward. Do the following steps to reach the period of the motion using action angle variables:

(i) Write down the Lagrangian of the particle and construct the Hamiltonian of the particle.

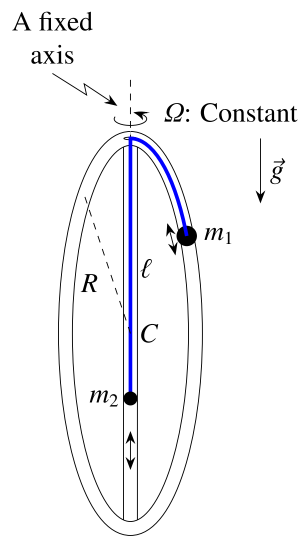
(ii) Compute the action variable J using part (i).

(iii) Calculate the frequency of oscialltion from part (ii) and then show that the period (T) is given by $T = 4\pi\sqrt{\frac{L}{g}}$.

Hint: Take advantage of the integral $\int \sqrt{1-\beta^2}\, d\beta = \frac{1}{2}\left(\beta\sqrt{1-\beta^2} + \arcsin\beta\right)$.

[AN-2018-Nov] Q2: A Bead on Spinning Hoop

A small bead of mass m_1 is arranged to slide on a thin vertical hoop of radius R (mass of the hoop is negligible). The bead is also connected to another object of mass m_2 by a massless rope of length $\ell > \pi R$ through a hole on the top point of the hoop as shown in the figure. The object of mass m_2 is allowed only to move vertically up and down. The hoop is rotated with a *constant* angular velocity $\boldsymbol{\Omega}$ about a vertical axis passing through the center C of the hoop. The motion takes place under uniform gravitational field $\boldsymbol{g}$ pointing downward. There is no friction anywhere in the system. The first part of the problem deals with finding the *tension* in the rope by using a Lagrange multiplier. Assume that the angular velocity of the hoop is such that $\Omega^2 = \dfrac{g}{\beta R}$ where β is a positive constant.



10 (a) Taking into account of the constraint related to the tension in the rope, choose suitable generalized coordinates and write down the Lagrangian of the sytem.

4 (b) Obtain the equations of motion by using the Lagrange equations modified with Lagrange multiplier.

10 (c) Determine the force of constraint as a function of the position of the bead and of given parameters of the system. Interpret physically.

(i) Find the position of the bead in which the force of constraint (the tension) you found in part (c) gets extremum if $\beta = 2$ is assumed.

(ii) Find the value of the tension at its extremum. Is this a maximum or minimum? Why?

14 (d) Let us go back to the equations of motion foun in part (b). Reduce them to a single second order differential equation by eliminating the force of constraint terms. Do the following analysis. Now take $m_1 = m_2 = m$ for simplicity.

(i) Obtain an algebraic equation for the equilibrium condition. Just get the equation, do not attempt to solve it.

(ii) Assuming a small disturbance from the equilibrium, make an expansion around the equilibrium to explore the motion around it from which define a so-called frequency of small oscillation in terms of equilibrium coordinate and given parameters.

(iii) Now if it is given that the equilibrium happens when the bead is at a distance $R/2$ from the xy plane, what would be the value of β. What is the frequency of oscillation?

6 (e) Simplify the Lagrangian in part (a) by eliminating the constraint and then construct the corresponding Hamiltonian, H, of the system. Write down Hamilton's equations.

6 (f) Discuss the following and in each case explain clearly why:

(i) Is the Hamiltonian a constant of motion?

(ii) Is the Hamiltonian equal to the total energy of the system?

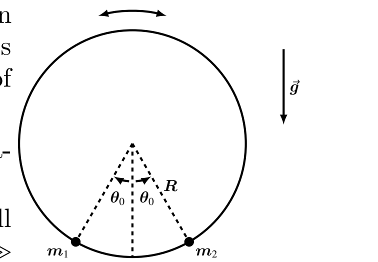
(iii) Is the total energy of the system conserved? Compute dE/dt in terms of the given parameters and generalized coordinate. Interpret your result.

4.2 AN-2018-1

[AN-2018-May] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

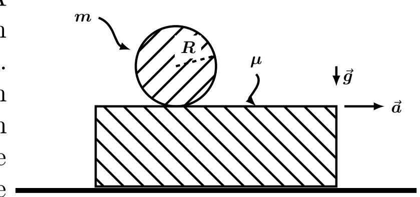
10 (a) There are two point particles m_1 and m_2 glued to a massless hoop of radius R. The masses make an angle θ_0 with the vertical as shown in the figure. The hoop is free to rotate in a vertical plane about an axis passing through its center. In the figure, the initial configuration of the system is shown.



(i) Write down the Lagrangian of the system and obtain the equation(s) of motion.

(ii) Determine the equilibrium angle and find the frequency of small oscillations about it. Discuss the cases (i) $m_1 \gg m_2$, (ii) $m_2 \gg m_1$, and (iii) $m_1 = m_2$.

10 (b) A platform is set in motion with a linear constant acceleration $\boldsymbol{a}$. A solid sphere of total mass m and radius R initially rests on the platform as shown in the figure under a downward uniform gravitational field $\boldsymbol{g}$. There is friction (with coefficient of friction μ) between the platform and the sphere. The purpose of the problem is to determine μ_{min} such that the sphere does a rolling without slipping motion by using the force of constraint information. Take the moment of inertia of the solid sphere as $I = \frac{2}{5}mR^2$.

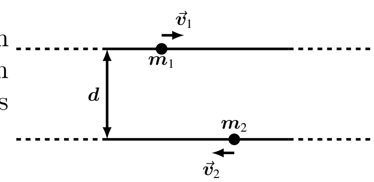


(i) Write down the Lagrangian of the system and find the equations of motion.

(ii) Using part i., express the force of the constraint in terms of the given parameters.

(iii) Using part ii., show that the minimum value of μ must be $\mu_{\text{min}} = 2a/7g$.

10 (c) Two non-interacting point particles m_1 and m_2 move toward each other with velocities $\vec{v}_1$ and $\vec{v}_2$ along two parallel straight lines which are a distance d apart as shown in the figure. The relative velocity is $\vec{v}_0$. Ignore the gravitational field.



(i) Express the total energy of the system with respect to the center of mass and define the effective potential U_{eff}. Express the angular momentum ℓ in terms of the given parameters.

(ii) Using the effective potential profile, discuss the possible motion for a given energy E.

10 (d) Consider the following transformation

$$Q = -q\left(q + \alpha\sqrt{q^2 + p}\right), \qquad P = -\left(q + \sqrt{q^2 + p}\right)$$

(i) For what value(s) of α does the transformation become canonical? Do this by using *two different methods.*

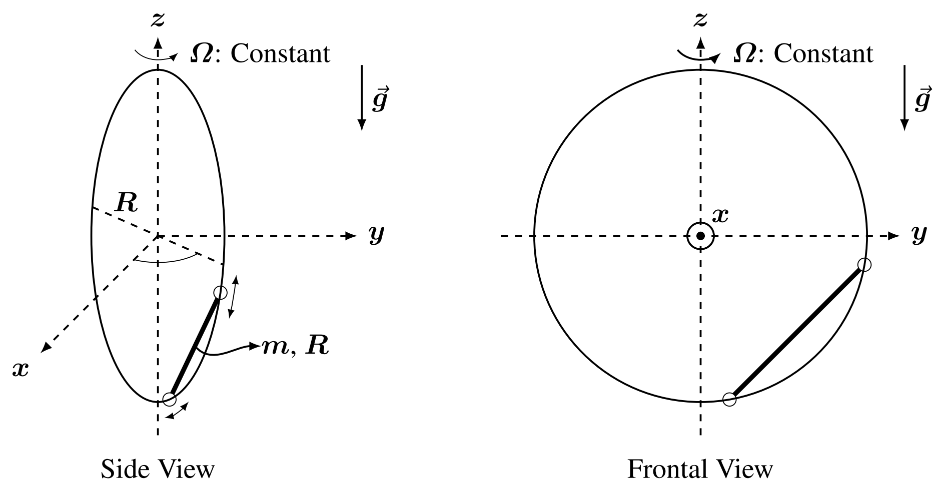
(ii) Find a suitable generating function.

10 (e) Consider two identical masses moving in one-dimension under the influence of the potential energy $U(z_1 - z_2)$ where $z_1(z_2)$ is the position of $m_1(m_2)$.

(i) Construct the Hamiltonian of the system and write down the Hamilton-Jacobi equation. Apply separation of variables.
Hint: Use the center of mass coordinates.

(ii) Find the solution to the Hamilton-Jacobi equation for $U = U_0\, e^{\beta(z_1 - z_2)}$.

[AN-2018-May] Q2: Dazzling Rod



A thin uniform rod of length R and mass m is attached by two massless rings at its ends to a vertical circular wire of radius R which is set in uniform rotation about a vertical axis passing through its center in a uniform gravitational field $\boldsymbol{g}$ pointing downward. In the figures above, the snapshots of the motion at two different instances are depicted for further clarification. There is no friction in the system.

4 (a) In the given coordinate system, find the translational kinetic energy of the center of mass of the rod, T_{trans}, and the potential energy of the system.

8 (b) The rotational energy of the rod, $T_{\text{rot}} = \frac{1}{2} I_{ij}\, \omega_i\, \omega_j$, would be rather subtle in the given configuration. Think of using the principal axes frame of the rod located at its center of mass. The number of terms in T_{rot} now reduces to three. The only thing that needs to be done is to express the angular velocity vector $\boldsymbol{\omega}$ of the rod (*not just the circular wire*) in the principal axes frame. If θ is the angle that the center of mass of the rod makes with the downward vertical direction, show that the rotational kinetic energy is

$$T_{\text{rot}} = \frac{1}{24} m R^2 \left(\dot{\theta}^2 + \Omega^2 \cos^2\theta \right) .$$

Using part (a), show also that the total kinetic energy of the system is

$$T = \frac{1}{24} m R^2 \left[10\, \dot{\theta}^2 + \Omega^2 \left(9 - 8 \cos^2\theta \right) \right] .$$

4 (c) Write down the Lagrangian of the system and obtain the Lagrange's equation(s) of motion.

8 (d) Alternatively, using the Euler's equation for rigid body motion (check the formula sheet), obtain the equation(s) of motion for the rod and compare with what you find in part (c).

6 (e) From the differential equation you get in part (c), determine the equilibrium configurations of the system by considering slow and fast rotations of the circular wire. Find the condition on Ω which results in a "non-trivial" equilibrium. For what values of Ω does that equilibrium occur at $\theta_0 = 60°$?

8 (f) Discuss the stability of the equilibria and find the frequency of oscillations when stable. For one of perturbed orbits that the center of mass of the rod traces around one of the stable equilibrium (equilibria), it is observed that it closes on itself after *two* oscillations in θ with *three* revolutions of the circular wire. Express the corresponding Ω in terms of the given parameters of the problem.

6 (g) Construct the Hamiltonian of the system and obtain the Hamilton's equations of motion. Compare with what you find in part (c) or part (d).

6

(h) Discuss the following and in each case explain clearly why:

(i) Is the Hamiltonian a constant of motion?

(ii) Is the Hamiltonian equal to the total energy of the system?

(iii) Is the total energy of the system conserved? Compute dE/dt in terms of the given parameters and generalized coordinates.

4.3 AN-2017-2

[AN-2017-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) A block of mass m_1 is free to slide along a straight horizontal rail. A simple pendulum of length ℓ and mass m_2 is attached to the block and the pendulum is restricted to swing in the vertical plane under an uniform downward gravitational field $\boldsymbol{g}$. There is no friction in the system. Using *the D'Alembert's principle only*, obtain the equations of motion.

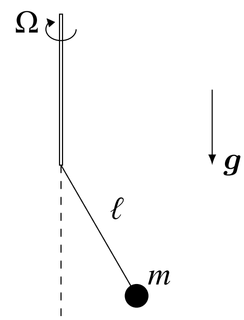
8 (b) A thin rod of length $2b$ and mass M moves on a smooth horizontal plane while one end of the rod is attached to a fixed point on the plane by a massless string of length R. By choosing appropriate generalized coordinates, write down the Lagrangian of the system. Obtain the equation(s) of the motion.

12 (c) A spherical pendulum of length ℓ and mass m which is kept in uniform rotation with Ω about the vertical massless shaft as shown in the figure. There is an uniform downward gravitational field $\boldsymbol{g}$. The purpose of the problem is to calculate the force of constraint associated with the uniform rotation.

(i) Write down the Lagrangian of the system.

(ii) Obtain the Lagrange's equation(s) of motion.

(iii) Calculate the force of constraint applied to maintain the uniform rotation.



10 (d) A particle of mass m is free to move in the xy-plane. There is another frame with coordinates X and Y rotating counterclockwise about the z-axis (z and Z axes are parallel to each other) with an angular speed $\omega(t)$. The origins of the frames coincide with each other. What is the Hamiltonian H in the xy-frame?

(i) Write down the transformation from (x, y) to (X, Y) both for coordinates and linear momenta (p_x, p_y) and (P_X, P_Y). Construct an F_2 type generating function.

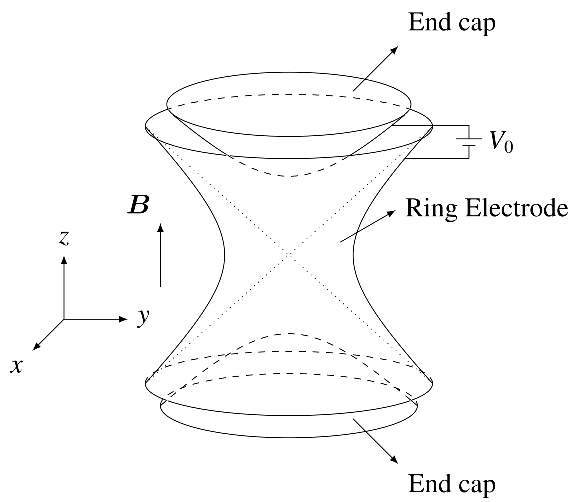
(ii) Find the transformed Hamiltonian K. Under what conditions is K conserved? Is $K = H$? Explain each and interpret physically.

(iii) Obtain the Hamilton's equations and interpret each term physically.

10 (e) There is a charged particle q with mass m is confined to xy-plane. A constant magnetic field $\boldsymbol{B}$ is applied perpendicular to the plane. By using the plane polar coordinates, construct the Hamiltonian of the system. Write down the Hamilton-Jacobi equation for the system and reduce the problem to the quadratures (i.e., obtain the solutions in the form of integrals).
Hint: Use $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r}$.

[AN-2017-Nov] Q2: Penning Trap

Confining a charged particle in a certain region of space has many applications in physics. The particle is placed in a region where there is an external electrostatic potential as well as a uniform external magnetic field. A sketch of the set up is given in the figure. There are three electrodes in the shape of hyperboloids of revolution: two end-cap electrodes and one ring electrode. A constant potential difference V_0 applied to the electrodes produces a non-uniform electric field in the space between the electrodes. The associated electrostatic potential is $\Phi_E(x,y,z) = \lambda_0(2z^2 - x^2 - y^2)$ where λ_0 is a constant parameter depending on V_0 and the geometry of the setup. There is also uniform magnetic field $\boldsymbol{B} = B\hat{z}$ whose effect can be given by a velocity dependent magnetic potential $\Phi_M(x,y,z) = -\dot{\boldsymbol{r}} \cdot \boldsymbol{A}$ with $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r}$. Such a device is known as the ideal Penning trap and in this problem we discuss the motion of a particle with positive charge q and mass m within the Penning trap. For later convenience let us define two parameters:



$$\omega_1 \equiv \sqrt{\frac{4q\lambda_0}{m}}, \quad \omega_2 \equiv \frac{qB}{m}.$$

6 (a) Write down the kinetic and potential energies of the particle by choosing the cylindrical coordinates (ρ, φ, z). Express the Lagrangian of the system.

2 (b) Obtain the Lagrange equations of the motion and show that the motion in z direction decouples from the motion in the other directions. Describe the motion in z direction.

4 (c) Does the system possess any symmetry? If so, what is the corresponding conserved quantity. Express it with the use of part (b) and interpret it physically.

20 (d) Since the motion in z is decoupled, let us concentrate on the horizontal motion. Using part (b),

(i) show that the plane motion is an effectively one-dimensional central force system.

(ii) In order to find an equilibrium radius ρ_0 giving a circular motion, what should be the condition on the parameters ω_1 and ω_2? Express ρ_0 in terms of the given parameters and constants of the problem.

(iii) Consider a small perturbation in the radial direction, obtain the frequency of small oscillations (ω_ρ). Express in terms of ω_1, ω_2.

(iv) Describe what types of motion take place in the horizontal plane. In order for the orbit to be closed after completing 2 angular revolutions after 7 radial oscillations, what should be the ω_2/ω_1 ratio?
Hint: You may want to compute the value of $\dot{\varphi}$ at the equilibrium $\rho = \rho_0$.

4 (e) Construct the Hamiltonian, H, of the system for the motion in the horizontal plane and determine the effective potential V_{eff} from the part of H (other than the radial energy).

10 (f) From V_{eff}, find its extremum as well as the radii where $V_{\text{eff}} = 0$. Determine the behavior of V_{eff} as $\rho \to 0$ and $\rho \to \infty$. Sketch V_{eff} qualitatively. Repeat these all for cases; (1) $\omega_2^2 > 2\omega_1^2$ and (2) $\omega_2^2 < 2\omega_1^2$. Which case is suitable for trapping the particle?

4 (g) For the suitable case above, what is the radius ρ_0 of the circular motion? Compare it with what you found in part (d). Obtain the frequency of small oscillations by using V_{eff} and again compare with what you got in part (d).

4.4 AN-2017-1

[AN-2017-May] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

8 (a) Determine the types of constraints (time-dependent/time-independent holonomic, nonholonomic etc.) in the systems defined below. When suitable, (a1) write down the constraint functions and (a2) give a set of generalized coordinates to define the motion of the systems in each case.

(i) A cylinder rolling without sliding on an inclinded plane.

(ii) A stick of length ℓ and mass m moves on a smooth horizontal plane with a constant acceleration $\boldsymbol{a}$ while it is being in uniform rotation (with angular frequency Ω) about a vertical axis passing through the center of mass of the stick.

(iii) Two point masses on a plane whose relative distance is known as $f(t)$.

(iv) A vertical disk rolling on a horizontal plane.

6 (b) A particle of mass m moves on the curve $z = h(x)$ under a gravitational field, $\boldsymbol{g} = -g\hat{z}$. Obtain the force of constraint(s) in magnitude by using the Lagrangian formulation *only*.

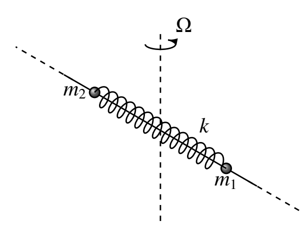
8 (c) (i) Suppose that there is a net force on a system of particles (with a total mass M) is $\boldsymbol{F}$ and in a fixed frame K' the position and velocity of the center of mass (CM) are $\boldsymbol{R}$ and $\boldsymbol{V}$, respectively. If $\boldsymbol{L}_S(\boldsymbol{N}_S)$ represents angular momentum(torque) in a frame $S = K'$ or CM, show that the following relations are hold :
(a) $\boldsymbol{L}_{K'} = \boldsymbol{L}_{\text{CM}} + M\boldsymbol{R} \times \boldsymbol{V}$, (b) $\boldsymbol{N}_{K'} = \boldsymbol{N}_{\text{CM}} + \boldsymbol{R} \times \boldsymbol{F}$, (c) $\dot{\boldsymbol{L}}_{\text{CM}} = \boldsymbol{N}_{\text{CM}}$.

(ii) Consider a system of particles with total mass M. The position vector of the center of mass of the system is $\boldsymbol{R}$ in a stationary frame K'. There is a moving point K whose position vector is $\boldsymbol{r}_K$. If $\boldsymbol{L}_K$ is the angular momentum of the system about the point K, show that

$$\dot{\boldsymbol{L}}_K = \boldsymbol{N}_K - M(\boldsymbol{R} - \boldsymbol{r}_K) \times \ddot{\boldsymbol{r}}_K$$

where N_K is the torque on the system with respect to the point K. Under what conditions the above formula reduces to the conventional one? Interpret this physically.

10 (d) Two beads of masses m_1 and m_2, which are free to slide along a massless wire on a *smooth horizontal* table, are connected by a massless spring with constant k as shown in the figure. The wire rotates with a constant angular velocity Ω about an axis shown.



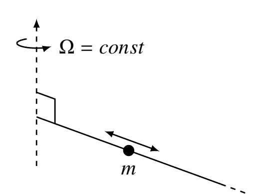
(i) By using the wire's frame (rotating frame) and the modified Newton's 2nd law ($m\,\boldsymbol{a}_r = \boldsymbol{F}_{eff}$), obtain the equations of motion for each bead.

(ii) Obtain the reaction forces on each bead.

(iii) Using the coordinates of the center of mass and the relative position, decouple the differential equations in part i. Find the positions of the beads as a function of time.

(iv) Express the total reaction force in terms of the coordinates in part iii.

10 (e) A bead of mass m is free to move on a massless *smooth horizontal* wire. The wire is in uniform rotation with constant angular velocity Ω about a vertical axis passing through one end of the wire as shown in the figure.



(i) Write down the Lagrangian of the system.

(ii) Construct the Hamiltonian and obtain the Hamilton-Jacobi (HJ) equation.

(iii) *Using only the HJ equation* found in part ii, find the position of the bead as a function of time.

8 (f) (i) Show that the volume elements in phase space are invariant under canonical transformations, that is, $dq_j dp_j = dQ_k dP_k$ if (q_j, p_j) pairs are related to (Q_k, P_k) pairs by a canonical transformation.

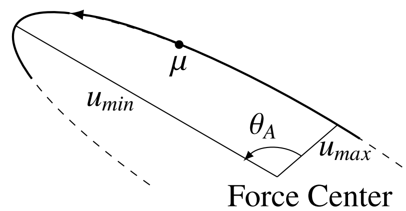
(ii) Consider the following transformation

$$Q_1 = q_1\,, \qquad Q_2 = p_2\,,$$
$$P_1 = p_1 + \alpha\, p_2\,, \quad P_2 = \beta q_1 - q_2$$

where α and β are some constants. To make the transformation canonical, determine the condition(s) on α and β.

(iii) Find a suitable generating function.
Hint: Consider using the exactness method with (p_1, q_2, Q_1, Q_2) as independent variables to obtain a mixed generating function

[AN-2017-May] Q2: Apside Down



A theorem in central force systems states that "*the **only** force laws yielding closed orbits for **all** bounded motions are the linear and inverse-square forces*". To show this, there are three main steps to follow: (1) finding potentials having constant apsidal angles (θ_A), the angle between the minimum and maximum position vectors from the force center (as shown in the figure), (2) expressing the apsidal angle for near-circular orbits, and finally (3) finding the potentials giving constant apsidal angles as rational multiple of π for general non-circular orbits. Let us go through the steps to achieve the goal. Consider the motion of a particle with reduced mass μ under the influence of a central force with potential energy $U(r)$.

2 (a) Write down the Lagrangian and Lagrange's equations of the particle.

2 (b) Find two constants of motion and explain in physical terms why they are expected to be conserved.

2 (c) Show that the motion can be described effectively in one-dimension under the influence of an effective potential energy $V_{\text{eff}}(r)$. Find $V_{\text{eff}}(r)$.

4 (d) Show that the radial equation can be put into the form

$$\frac{1}{2}\bar{\mu}\left(\frac{du}{d\theta}\right)^2 + V_{\text{eff}}(u) = E\ ,$$

where E is the total energy and $u \equiv 1/r$. Find $\bar{\mu}$.

3 (e) Derive the condition for a circular orbit at $u = u_0 = \text{const.}$ by using $V_{\text{eff}}(u)$ found in part (d).

6 (f) Now assume a small perturbation around the equilibrium $u = u_0$ such that $u = u_0 + \epsilon\eta(\theta)$ (ϵ is the control parameter). Express the energy E in part (d) by expanding $V_{\text{eff}}(u)$ around u_0. Keep the terms up to and including $\mathcal{O}(\epsilon^2)$.

6 (g) It is known that the solution of a one-dimensional simple harmonic oscillator (SHO) is $x(t) = A\cos(\omega t)$ with the total energy $E_{\text{SHO}} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ with $\omega^2 = k/m$. Comparing the energy E in part (f) with E_{SHO}, write down the solution $u(\theta)$. Find Ω in terms of $U(u)$ and its derivatives.

3 (h) Using the definition of the apsidal angle (θ_A), show that $\theta_A = \pi/\Omega$.

6 (i) The arguments so far are valid for only small deviations from the circular orbit. To get a constant θ_A for any value of E, Ω should be a positive constant for any distance r. Then determine the form of $U(r) = c\,r^d$, namely express c and d in terms of the parameters. Find the apsidal angle θ_A.

8 (j) For $d > 0$, consider the effective potential $V_{\text{eff}}(u)$ and sketch it as a function of u. By using the limiting form of the energy expression in part(d) for very large E values and with the help of the SHO analogy determine θ_A. From the form of θ_A in part (i), find the value of d and the potential $U(r)$. What type of force does the potential energy $U(r)$ correspond to?

8 (k) For $d < 0$, sketch the effective potential $V_{\text{eff}}(u)$. In this case consider $E \to 0$ limit in the energy expression. By making a substitution, put the equation into the SHO form and identify the angular frequency. Obtain θ and the value of d. What type of force does the potential energy $U(r)$ correspond to? Comment on $d = 0$ case.

4.5 AN-2016-2

[AN-2016-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

6 (a) A particle of mass m is making a straight-line motion, say along the y axis, with a displacement of $\Delta y = y_2 - y_1$ in a $\Delta t = t_2 - t_1$ time interval. The time averaged kinetic energy of the particle is

$$\langle T \rangle = \frac{1}{\Delta t} \int_{t_1}^{t_2} \frac{1}{2} m \dot{y}^2 dt \ ,$$

where $\dot{y}$ is the time derivative of y. Determine the position of the particle, $y(t)$, as a function of time in terms of the given parameters so that the average kinetic energy $\langle T \rangle$ gets its *minimum* value..

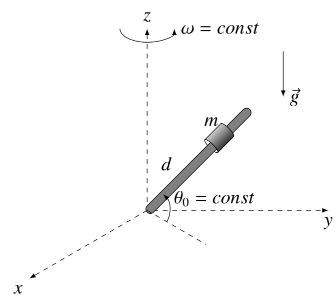
10 (b) A particle of mass μ moves in a central force system with a potential energy $U(r) = \kappa r^4$.

(i) For the particle to have a circular orbit of radius r_0, what angular momentum ℓ and total energy E should it have? Express them in terms of the given parameters.

(ii) Find the condition on κ for the mass μ to have a *stable* circular orbit at r_0.

(iii) If a very small radial kick is given to the particle at r_0, determine whether the orbit of the subsequent motion is *closed* or not. Explain.

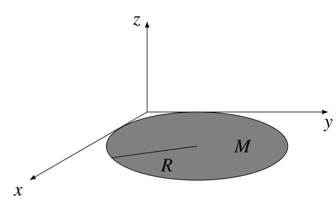
8 (c) There is a bead of mass m along a massless rod which makes a constant angle θ_0 with the horizontal plane. The bottom end of the rod is fixed at the point O on the horizontal plane, and the rod is kept rotating uniformly with angular velocity ω about the vertical axis passing through O. The system is under a uniform gravitational field $\boldsymbol{g} = -g\hat{z}$ and there is no friction.



(i) By using the rotating frame and the modified Newton's 2nd law ($m\boldsymbol{a}_r = \boldsymbol{F}_{eff}$), determine the value of ω to keep m at rest at a distance d. Express also the normal force in terms of the given parameters.

(ii) Repeat the above part by applying the Newton's 2nd law in the inertial (fixed) frame.

8 (d) A thin uniform disc of radius R and total mass M lies on a horizontal surface.



(i) Calculate the inertia tensor J_{ij} of the disc in the given coordinate system.

(ii) If the disc is rotated about the z axis with a uniform angular velocity ω, calculate the kinetic energy of the disc.
Hint: Think of using the parallel axis theorem.

10 (e) Consider the following transformation from $(q, p) \to (Q, P)$,

$$Q = q^\alpha \cos(\beta p) \ , \quad P = q^\alpha \sin(\beta p) \ ,$$

where α and β are some constants.

(i) Determine the values of α and β so that the transformation is canonical.

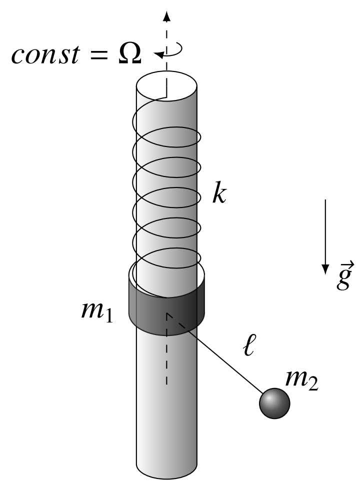
(ii) Find a generating function of type F_3.

8 (f) There is a particle of mass m falling under a uniform gravitation field $\boldsymbol{g}$.

(i) Write down the Hamilton-Jacobi equation for the system.

(ii) Determine the motion of the particle by solving the Hamilton-Jacobi equation in part (i).

[AN-2016-Nov] Q2:



A pointlike object of m_1 is constrained to slide along a vertical massless shaft. One end of a massless spring of constant k is connected to m_1 while the other end is attached to the top of the shaft. A massless string of length ℓ connects m_1 to a pointlike object of mass m_2 as shown in the figure. The shaft is set in uniform rotation with angular speed Ω. The system is in a uniform gravitational field $\boldsymbol{g}$. There is no friction anywhere in the system. Ignore the relaxed length of the spring.

12 (a) Write down the Lagrangian of the system by choosing suitable generalized coordinate(s).

4 (b) Obtain the Euler-Lagrange equation(s) of motion.

6 (c) Using part (b), determine the equilibrium configuration(s) if there is(are) any. Interpret your result physically.

In the rest of the problem, assume that $k \to \infty$, that is, the spring is infinitely stiff.

3 (d) Reduce the equation(s) in part (b). Interpret your result physically.

8 (e) Using part (d), discuss the stability of the equilibrium(s) by making an expansion around it/them. Determine the condition(s) on Ω from the stability point of view. Find the frequency of small oscillations whenever relevant.

8 (f) Construct the Hamiltonian of the system and obtain the Hamilton's equations of motion.

3 (g) Show that from the Hamilton's equations of motion, the equation(s) found in part (d) follow(s).

6 (h) Discuss the following and in each case explain why:

(i) Is the Hamiltonian a constant of motion?

(ii) Is the Hamiltonian equal to the total energy of the system?

(iii) Is the total energy of the system conserved? Compute dE/dt in terms of the given parameters and generalized coordinates.

5. Statistical Mechanics

5.1 SM-2018-2

[SM-2018-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) Provide **brief** answers to the following conceptual questions. Use formulae, graphs and text as you see fit as long as it is sufficiently accurate.

(i) (2 points) For a gas of diatomic molecules, which can be considered otherwise ideal, roughly estimate the temperature at which the rotational degrees of freedom get activated if their moment of intertia is I.
Hint : Here "activated" means "start to contribute significantly to the C_V*".*

(ii) (2 points) Describe the concept of chemical potential. Do you think there could be systems with zero chemical potential? What kind of systems would these be?

(iii) (2 points) "*Negative temperature* may occur in systems with bounded entropy." Prove or justify this statement.

(iv) (2 points) The equation of state for a cavity filled with photons is $U = 3PV$, which implies that the photons have nonzero pressure. Explain the origin of this pressure.

(v) (2 points) The Hamiltonian of a classical anharmonic oscillator is given by

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 + \lambda q^4,$$

where λ is a positive constant. Is this system expected to obey the *equipartition theorem*? Explain your answer.

10 (b) Two large heat reservoirs are available at 900 K (H) and 300 K (C).

(i) (3 points) 100 J of heat is removed from reservoir H and added to reservoir C. What is the entropy change of the total system, i.e. the two reservoirs together?

(ii) (3 points) A reversible engine operates between H and C. If 100 J of heat is removed from H, how much work is done by the engine and how much heat is added to reservoir C?

(iii) (4 points) What is the entropy change of the total system (engine + reservoirs) in the process described in part(ii)?

10 (c) The energy dispersion relation of ultrarelativistic particles is given by

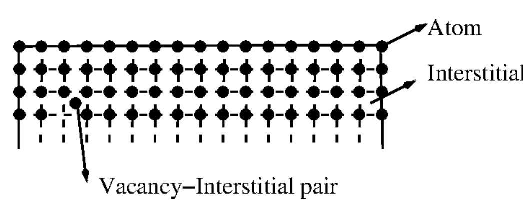
$$\varepsilon(k) = \hbar ck.$$

where c is the speed of light and k is the norm of the wavevector.

(i) (5 points) Calculate the three-dimensional density of states $D(\varepsilon)$ for a noninteracting gas of such particles, occupying a volume V.
Hint: $D(k) = Vk^2/2\pi^2$.

(ii) (5 points) For N ultrarelativistic, spin-$\frac{1}{2}$ fermions with this dispersion relation, determine the Fermi energy in terms of the particle density.

10 (d) Consider an idealization of a crystal which has N lattice points and N interstitial positions (where atoms can reside) as seen in the figure. Let E be the energy necessary to remove an atom from a lattice site to an interstitial position and let n be the number of atoms occupying interstitial sites in equilibrium.



(i) (2 points) What is the internal energy of the system as a function of n?

(ii) (3 points) What is the entropy S? Give an asymptotic formula valid for $n \gg 1$.

(iii) (5 points) In equilibrium at temperature T, using the fact that the Helmholtz free energy is minimized at equilibrium, derive how many such defects there are in the solid at a given temperature T, i.e. find n.

10 (e) A reversible cyclic engine is constructed from an ideal gas using the following processes:

1. From $A = (P_1, V_1)$ to $B = (P_2, V_2)$, adiabatic expansion
2. From $B = (P_2, V_2)$ to $C = (P_2, V_1)$, isobaric (same pressure) compression ($V_2 > V_1$)
3. From $C = (P_2, V_1)$ to $A = (P_1, V_1)$, isochoric (same volume) process

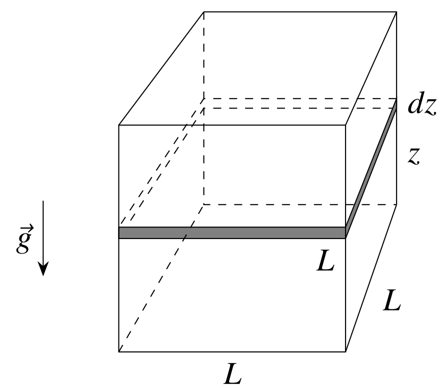
Answer the following questions:

(i) (1 points) Sketch this process on a $P - V$ graph.

(ii) (2 points) Is this a Carnot engine? How did you make that decision?

(iii) (7 points) Calculate the efficiency of the cycle. You can give your answer in terms of P_1, P_2, V_1, V_2 and γ.
Hint: Calculate the work and the heat absorbed.

[SM-2018-Nov] Q2: Ideal gas in the presence of gravity



A cubic box of dimensions $L \times L \times L$ is filled with an ideal, noninteracting gas of spinless particles of mass m. Answer the following questions :

4 (a) Start out by ignoring the gravitational force and treating the system quantum mechanically. Write down the Hamiltonian and the corresponding energy eigenstates.

3 (b) Write down the partition function.

8 (c) Calculate the low and high temperature limits of the partition function. Justify the approximations you make.
Hints:

1. *In the low temperature limit, keep only the* ***two*** *largest terms of the partition function.*
2. *In the high temperature limit, convert the sums into integrals. You will have to convert the lower limit of this integral to zero to be able to integrate it. Justify also this conversion.*

8 (d) Find C_V in both limits. Is the result consistent with the equipartition theorem in the high-temperature limit? Explain.

3 (e) Next, introduce gravity into the problem. Write down the classical Hamiltonian of the particles in the box.

8 (f) Calculate the partition function by means of employing a phase-space integral.

8 (g) Calculate C_V in the presence of gravity and show that it reduces to your previous result as $g \to 0$.

8

(h) Determine $P(z)/P_0$ where $P(z)$ is the pressure as a function of height and $P_0 \equiv P(z = 0)$ is the pressure at the bottom of the box. Use the following steps in your derivation

1. Utilizing the figure above, write an expression for $dP(z) = P(z + dz) - P(z)$ for a sliver of gas of thickness dz at a height z. Your answer should be in terms of m, g, L, dz and $\rho(z)$ where $\rho(z)$ is the volume density of the gas at a height of z.
2. Combine your answer from the previous step with the fact that $P(z) = \rho(z)/\beta$ ($\beta = kT$) and integrate dP to find $P(z)$.
3. Finally, make sure that your answer makes sense when $g \to 0$.

5.2 SM-2018-1

[SM-2018-May] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

8 (a) The cosmic microwave background radiation has a blackbody spectrum at $T = 2.7$ K. Provide a rough estimate of the number of photons due to this radiation in space in a volume of $V = 1$ cm^3. You will need

$$\hbar c \approx 2 \times 10^{-7} \text{ eV} \cdot \text{m} ,$$
$$kT \approx 2.25 \times 10^{-4} \text{ eV} ,$$
$$\int_0^\infty dx \frac{x^2}{e^x - 1} \approx 2.404$$

8 (b) Consider an ideal gas:

(i) Write down the internal energy and the equation of state for this ideal gas.

(ii) Starting from the differential form of the internal energy as given by the first law of thermodynamics, calculate the entropy.

(iii) Using this formula calculate the change in entropy of free expansion where the volume of a thermally isolated gas whose temperature is initially T_0 is suddenly increased from V_0 to $2V_0$. The final entropy is measured after the gas has found equilibrium again.

14 (c) A reversible cyclic engine is constructed from an ideal gas using the following process steps:

1. from $a = (P_1, V_1)$ to $b = (P_2, V_2)$, adiabatic expansion, $(V_1 < V_2)$,
2. from $b = (P_2, V_2)$ to $c = (P_2, V_1)$, isobaric (same pressure) compression, $(V_2 > V_1)$,
3. from $c = (P_2, V_1)$ to $a = (P_1, V_1)$, isochoric (same volume) process.

Answer the following questions.

(i) Sketch this process on a $P - V$ graph.

(ii) Is this a Carnot engine? How did you make that decision?

(iii) Calculate the work done by the system during this process.

(iv) Calculate the heat taken in by the system during this process.

(v) What is the efficiency of this cycle?

8 (d) A spin 3/2 system has the following energy levels corresponding to each value of the spin quantum number I_z:

$$E(I_z) = \begin{cases} \epsilon_1 & \text{for } I_z = -3/2 , \\ \epsilon_2 = \epsilon_1 + \delta & \text{for } I_z = -1/2 , \\ \epsilon_3 = \epsilon_1 + \Delta & \text{for } I_z = +1/2 , \\ \epsilon_3 & \text{for } I_z = +3/2 . \end{cases}$$

where the last two states have the same energy and $\epsilon_1 < \epsilon_2 < \epsilon_3$.

(i) Calculate the thermal average of the spin, $\langle I_z \rangle$, at an arbitrary temperature T.

(ii) Without using the partition function, argue that $C_V \to 0$ as $T \to \infty$, (i.e., explain qualitatively).

8 (e) A system has three single-particle energy states, two of which are degenerate: $E_1 = E_2 = \varepsilon, E_3 = 2\varepsilon$. Two particles are placed into the system. In other words, $N = 2$ particles are distributed to these 3 states. Write down the canonical partition function of this system if the particles are

(i) distinguishable,

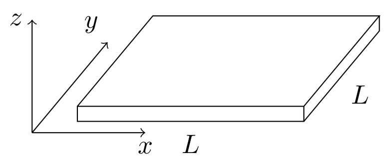
(ii) fermions,

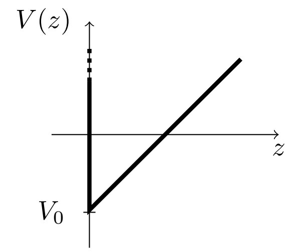
(iii) bosons.

4 (f) Why do we use the prefactor $1/h$ in *classical phase-space integrals* even though one usually associates this fundamental constant with quantum mechanics? Explain briefly.

[SM-2018-May] Q2: Adsorbates on a surface

A simple model for adsorbates (molecules attached to a surface) can be constructed as follows:





1. Assume that the molecules can move freely on the surface, which lies on the xy-plane and are considered to be particles in a box (PIB). The side length of the square material is L. The mass of the particles is m.
2. The molecules are trapped along the z axis by a triangular potential well, whose potential energy is given by

$$V(z) = \begin{cases} \infty, & z < 0 \\ \alpha z - V_0, & z \geq 0 \end{cases}$$

as seen in the figure. Here, the surface is assumed to be located at $z = 0$.

Answer the following questions:

4 (a) Write down the quantum mechanical Hamiltonian for this problem.

5 (b) The solutions of the triangular potential are the so-called Airy functions with approximate eigenvalues:

$$E_n = \sqrt{\left(\frac{\hbar^2}{2m}\right)} \left[\frac{3\pi\alpha}{2}\left(n - \frac{1}{4}\right)\right]^{2/3} - V_0 \qquad n = 1, 2, 3\cdots$$

Write down the energy eigenvalues of the Hamiltonian constructed in part (a).

5 (c) Write down the total partition function, which is separable:

$$Z = Z_{PIB} \times Z_{tri}$$

5 (d) Let us look at the low temperature limit of the partition function. Keep only the lowest energy eigenvalue of $V(z)$ in Z_{tri}. How low must the temperature be for this approximation to be valid? Similarly, convert the sums in Z_{PIB} to integrals. Justify this approximation also and evaluate the integral.

7 (e) Using Z as the starting point, calculate C_V in this low temperature limit.

4 (f) Next, let us look at the high-temperature range. In the high-temperature range, the classical mechanical description becomes valid. Write down the classical Hamiltonian.

7 (g) Calculate the classical partition function.

7 (h) Find C_V as $T \to \infty$.

6 (i) Using the quantum mechanical and classical results, sketch C_V at all temperature ranges. Of course, it will be approximate. Do not worry about possible peaks in the midrange. Just assume it is monotonic.

5.3 SM-2017-2

[SM-2017-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) **Conceptual/Short questions:** Provide **brief** answers to the following conceptual questions. Use formulae, graphs and text as you see fit as long as it is sufficiently accurate.

(i) Describe the ultraviolet catastrophe.

(ii) Why can't a Carnot engine have 100% efficiency?

(iii) Justify **briefly** the Boltzmann definition of entropy

$$S = k_B \ln \Omega$$

by answering the following questions: What is Ω? Why does it make sense for entropy to be proportional to $\ln \Omega$?

(iv) Why do we use the Helmholtz free energy in problems that involve a canonical ensemble?

5 (b) **Heat Capacity of a Metal:** Show that, if the entropy of a metal at low temperatures grows linearly with temperature, the heat capacity, C_V also grows linearly in the same range.

10 (c) **Particles in a Box:** The energy levels of particles in a one-dimensional box are given by

$$\varepsilon_n = \frac{\pi^2 \hbar^2}{2mL^2} n^2 \equiv \alpha n^2.$$

(i) Calculate the canonical partition function for the low temperature range ($\alpha \gg k_B T$).
Hint: Keep the first term in the sum.

(ii) Calculate the internal energy, U in this limit.

(iii) Calculate the canonical partition function in the high temperature limit ($\alpha \ll k_B T$).
Hint: Approximate the sum with an integral.

(iv) Calculate the internal energy, U, in this limit.

15 (d) **Classical Particle in a Trap:** A point particle with mass m moves along one axis under the effect of a conservative force with the following potential

$$V(q) = \begin{cases} \frac{1}{2} m\omega^2 (q+a)^2 \,, & q \le -a \\ 0 \,, & -a < q < a \\ \frac{1}{2} m\omega^2 (q-a)^2 \,, & q \ge a \end{cases}$$

where a is a positive constant.

(i) Sketch the potential as a function of q.

(ii) Calculate the N-particle canonical partition function, Q_N, if the particles are distinguishable (e.g. localized in well-separated traps).

(iii) Find the average internal energy.

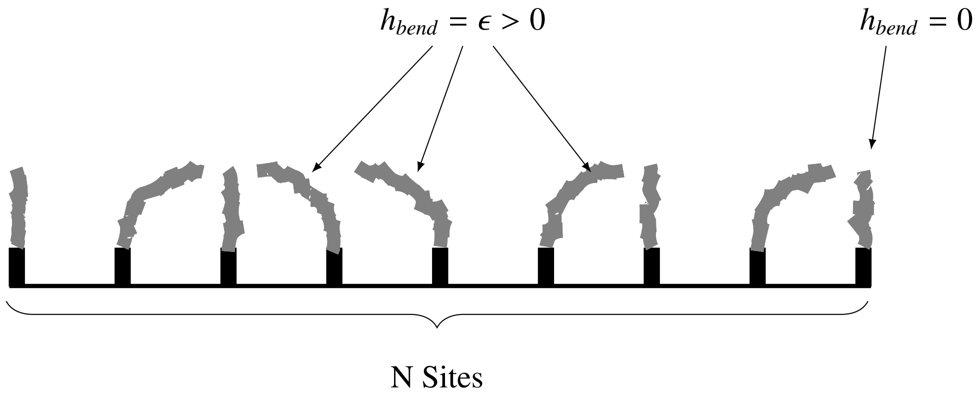
(iv) When $a \to 0$, does average internal energy result reduce to that of a regular harmonic oscillator?
Hint: You may make use of the equipartition theorem.

10 (e) **Fermi Energy in Two Dimensions:** A non-relativistic collection of noninteracting electrons is usually referred to as a free electron gas. A two-dimensional free electron gas can approximate the narrow layer of electrons that accumulates at the interface of certain heterostructures.

(i) What is the density of states, $D(\varepsilon)$, for a free particle in two dimensions, occupying a total area A?

(ii) Derive the Fermi energy in terms of area A and the number of particles N in a two-dimensional, non-relativistic Fermi gas at zero temperature.

[SM-2017-Nov] Q2: Chain of Upright Polymers



In this question, we consider a one-dimensional chain of $N \gg 1$ upright polymers, each localized on a lattice site. Each polymer may be in two energy states: it can be straight (with energy $h_{bend} = 0$) or it can bend (to the right or to the left) with energy $h_{bend} = \varepsilon > 0$, regardless of the bending direction (See Figure). Answer the following questions:

5 (a) Find the total number of microstates $\Omega(m, N)$ for a total energy of $E = m\varepsilon$. Here m is the number of bent polymers with $m \gg 1$.
Hint: You must also take care of the degeneracy associated with the bending direction.

8 (b) Calculate the entropy with the help of the Stirling approximation.

10 (c) Determine the temperature as a function of m and from the resulting expression, calculate m as a function of T.

7 (d) Determine the internal energy as a function of T.

10 (e) Calculate the heat capacity, C.

5 (f) Determine the behavior of C in the high-temperature and low-temperature limits.

5 (g) This question was solved by treating the system as a part of a microcanonical ensemble. Decribe briefly how you would approach the same problem treating the system as a part of a canonical ensemble. Do you expect to obtain the same results?

5.4 SM-2017-1

[SM-2017-May] Q1: Answer the Following Questions.

10 (a) Unlike the ideal gas that is assumed to be made up of noninteracting point particles, real gas atoms/molecules have a finite volume and interact. A more realistic description of a gas is the van der Waals model. In this model, the equation of state and the internal energy are

$$\left[P + a\left(\frac{n}{V}\right)^2\right]\left(\frac{V}{n} - b\right) = RT \ ,$$

$$U = \frac{3}{2}nRT - \frac{n^2 a}{V}$$

where P, V and n are the pressure, volume and number of moles, respectively. The constants a and b are positive values that model finite size of atoms and finite inter-particle interaction. Calculate the entropy change during an isothermal process that occurs at $T = T_0$ where the final volume is twice the initial volume, V_0.

10 (b) Two identical blocks of copper, one at T_1 and the other at T_2 are placed in thermal contact with each and are thermally isolated from everything else. Given that the heat capacity at constant volume of each block, C, is independent of temperature, obtain an expression for the change in entropy when the system reaches equilibrium with respect to the initial state. Also, show that $\Delta S > 0$ regardless of whether $T_1 > T_2$ or $T_2 > T_1$. (Assume $T_1 \neq T_2$.)
Hint: The arithmetic-geometric mean inequality states that the geometric mean is always smaller than the arithmetic mean: $\sqrt{xy} \leq (x + y)/2$.

10 (c) A one-dimensional quantum harmonic oscillator (whose ground state energy is $\frac{\hbar\omega}{2}$) is in thermal equilibrium with a heat bath at temperature T. What is the mean value of the oscillator's energy $\langle E \rangle$ as a function of T?

10 (d) Consider a system of N non-interacting particles ($N \gg 1$) in which the energy of each particle can assume only two values, $\varepsilon_0 = 0$ and $\varepsilon_1 = E$ where ($E > 0$). If in a particular macrostate, the occupation number of the state with label ε_0 is n_0, find the temperature, T, of the system as a function of the total internal energy U.

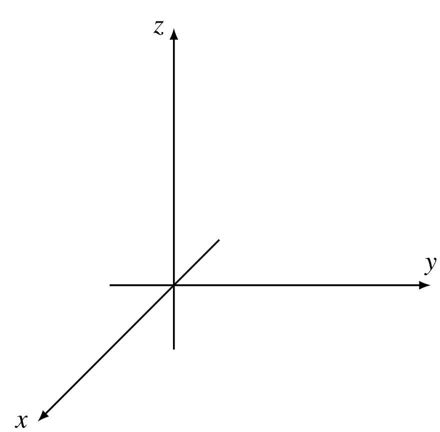
10 (e) In our three-dimensional universe, the energy density of black body radiation depends on the temperature as T^α where $\alpha = 4$. What is the value of α in an n-dimensional universe?
Hint: You will end up with a complicated integral over ω. *You can extract the temperature dependence without actually evaluating the integral.*

[SM-2017-May] Q2: Dipolar molecules

An ideal classical gas is formed by N indistinguishable non interacting diatomic molecules. Each one of the has an electric dipole moment of magnitude D. The whole gas is in thermal equilibrium at temperature T and is under the effect of a constant electric field with intensity $\mathcal{E}$ directed along the z axis. The Hamiltonian of a single dipole is

$$\mathcal{H} = \frac{1}{2I}p_\theta^2 + \frac{1}{2I\sin^2\theta}p_\phi^2 - D\mathcal{E}\cos\theta$$

where I is the moment of inertia of the molecule, (θ, ϕ) are the spherical polar angles to represent the orientation and (p_θ, p_ϕ) are the associated momenta. The Hamiltonian therefore contains contributions from the rotational degrees of freedom of the molecules as well as the coupling of the dipoles with the electric field.



2 (a) Sketch the vectors $\vec{D}$ and $\vec{\mathcal{E}}$ on the given axis in the figure and mark the angles clearly.

2 (b) In the analytical mechanics exam that you took yesterday, you found out how the phase space volume element dv changes under a canonical coordinate transform. For this question, in particular, going from Cartesian to spherical coordinates in phase space

$$(x, y, z, p_x, p_y, p_z) \to (r, \theta, \phi, p_r, p_\theta, p_\phi)$$

causes the volume element to transform as

$$dv = dxdydzdp_xdp_ydp_z \to dv = drd\theta d\phi dp_rdp_\theta dp_\phi$$

and since, in a rotational Hamiltonian, the radius does not change, you only need

$$dv = d\theta d\phi dp_\theta dp_\phi.$$

Using this information, write down the classical partition function for a single particle. Do not perform any integrations at this stage.

10 (c) Perform the integrations in the previous part to prove that the partition function of a single dipole is given by

$$Q_1 = \frac{2I \sinh(\beta D\mathcal{E})}{\hbar^2 \beta^2 D\mathcal{E}}$$

Hint: $$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \quad .$$

2 (d) Write down the N-particle partition function, Q_N.

6 (e) Calculate the specific heat, C_V.

6 (f) Calculate the high temperature limit of C_V.

6 (g) When $\mathcal{E} \to 0$, the system reduces to a regular collection of diatomic molecules. In this limit, does the specific heat yield the result you would expect from the equipartition theorem? Explain briefly.

8 (h) Macroscopic polarization is defined as

$$P = \frac{N}{V} \langle D \cos\theta \rangle.$$

Starting from Q_N derived above, show that

$$P = \frac{N}{V} \left(D \coth(\beta D\mathcal{E}) - \frac{1}{\beta\mathcal{E}} \right)$$

8 (i) In the limit of weak field $\beta D\mathcal{E} \to 0$, show that the dielectric constant defined by

$$\epsilon\mathcal{E} = \epsilon_0\mathcal{E} + P$$

is equal to $\epsilon = \epsilon_0 + \frac{N\beta D^2}{3V}$.

5.5 SM-2016-2

[SM-2016-Nov] Q1: Answer the Following Questions.

Note: The individual parts of the following question are intended to be independent from each other.

10 (a) For most of the gases at room temperature, the vibrational modes are frozen and the translational and rotational degrees of freedom can be treated classically. Using the equipartition theorem, determine the molar heat capacity of

(i) Ne gas,

(ii) CO_2 gas (note that CO_2 molecules are linear), and

(iii) H_2O gas.

10 (b) Consider a photon gas inside a cavity of volume V and temperature T. The volume of the cavity is then expanded to volume $2V$ in such a way that (1) the expansion is so quick that there is no appreciable energy (heat) transfer from the walls of the cavity to the photon gas and (2) the walls move at non-relativistic speeds so that the change is "slow" for the photon gas.

(i) Discuss if this is an example of an adiabatic process for the photon gas.

(ii) Find the final temperature of the gas.

Hint: The Helmholtz free energy of the photon gas is given as $F(T,V) = -bT^4V$ *where* b *is some constant.*

10 (c) N independent and distinguishable point particles move in a one-dimensional domain between $q = 0$ and $q = L$. Determine the one-dimensional pressure in the whole system, if the single-particle Hamiltonian is given by

$$\mathcal{H} = \mathcal{H}(p,q) = \frac{p^2}{2m} - \alpha \ln\left(\frac{q}{L_0}\right) , \qquad (\alpha > 0).$$

Assume that the system is in a canonical ensemble. In the above expression, α is a constant giving the strength of the potential and L_0 is a characteristic length scale. What is the low-temperature limit of the pressure?

10 (d) In a Fermi gas of N spin $s = 1/2$ particles with mass m, the particles occupy a two-dimensional domain with an area A. If the temperature is T, determine the Fermi energy, ε_F, as a function of particle density, n. (Assume $k_BT \ll \varepsilon_F$.)

10 (e) The *Joule expansion* of a gas is the sudden expansion of the gas from an initial volume V_i to a final volume V_f. The expansion is so fast that there is not enough time for heat absorption (as a result it can be assumed that the gas is thermally insulated). After that, the gas is allowed to equilibrate. Joule's empirical work led to the observation that the temperature T of the gas does not change during and after the expansion process.

Assume that the gas obeys the ideal gas law. Answer the following questions:

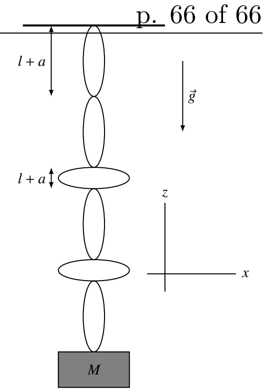
(i) Give a plausible explanation for the fact that the temperature doesn't change.

(ii) Why is this an irreversible process?

(iii) Can the entropy change $\varDelta S$ between the initial and final states be calculated using a process like this?

(iv) If $V_f = 2V_i$, calculate the entropy change during the expansion.

[SM-2016-Nov] Q2: Hanging Chain



A one dimensional chain, made of massless rings, hangs from a ceiling. One of its extremes is fixed, while the other holds a mass M as shown in figure. Gravity acts along the negative z direction. The chain is formed by two kinds of rings: they are ellipses with the major axis oriented vertically or horizontally. The major and minor axes have lengths $(l + a)$ and $(l - a)$, respectively. Although the number of the rings is fixed, the rings are allowed to change orientation (vertical to horizontal) in accordance with the finite temperature. The number of rings in the vertical direction for a given state of the chain is n.

2 (a) Write down the total length, L, of the chain in terms of n and the other relevant constants.

4 (b) What is the internal energy of the chain for a given n?

4 (c) For a given length L, determine the number of possible microstates, $g(n)$. Explain your answer briefly.

6 (d) Using g_n and assuming a canonical distribution, write down and *simplify* the partition function, Q_N, for the entire chain. Note that the rings are distinguishable.
Hint 1: You will have to assign energies to the vertical and horizontal orientations.
Hint 2: Depending on your approach, you may or may not need the binomial expansion,

$$(x + y)^M = \sum_k \binom{M}{k} x^k y^{M-k}.$$

10 (e) Using Q_N as the starting point, derive the entropy of the system. Find the high and low temperature limits.

10 (f) Calculate the average length, $\langle L \rangle$. Find the limit of the length at the high and low temperature limits.

10 (g) Show that the linear response at the high temperature limit, which is defined as the change in the average length as a function of the force $F = Mg$,

$$\chi \equiv \frac{\partial \langle L \rangle}{\partial F} = \frac{Na^2}{kT}.$$

4 (h) This result formally resembles the magnetic susceptibility

$$\chi = \frac{\partial \langle \mathcal{M} \rangle}{\partial H} = \frac{N\mu^2}{kT}$$

of a one-dimension system of up and down spins under an external B-field. This is the well-known Curie law and $\mathcal{M}$ is the total magnetization of the system. Explain the resemblence by drawing analogies to μ and H in the current problem of the hanging chain.