Static and stationary Lifshitz black holes and their conserved Killing charges

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D.O. Devecioğlu, ÖS, Phys. Rev. **D 83** (2011) 021503(R), arXiv:1010.1711 [hep-th];
D.O. Devecioğlu, ÖS, Phys. Rev. **D 83** (2011) 124041, arXiv:1103.1993 [hep-th];
ÖS, Phys. Rev. D **84** (2011) 127501; arXiv:1109.4721 [hep-th].

Lifshitz spacetimes:

• emerged in applications of the AdS/CFT duality to non-relativistic, specifically, condensed matter systems.

• thought of as gravity duals of theories with nontrivial scaling properties.

Lifshitz black holes:

• BH solutions to some gravitational (+ matter) theory that are asymptotically Lifshitz.

• needed for describing "finite temperature aspects" of these non-relativistic systems.

• nice review on all related story on AdS/CMT by S.A. Hartnoll, Class. Quant. Grav. **26**, 224002 (2009), arXiv:0903.3246 [hep-th].

Motivations:

• There were few exact analytic Lifshitz BHs, and for many of these the gravitational energy computation (+ a study of the 1st law of BH thermodynamics) was missing.

• All known exact analytic Lifshitz BHs are static. Q: Are there any stationary ones?

Advertisements:

• The first two papers have provided an answer to the first problem.

• The third paper is the first work to present stationary D-dimensional exact analytic Lifshitz spacetimes and (similarly) black "objects".

A closer look at *static* Lifshitz spacetimes:

• a typical feature in condensed matter systems is the "dynamical scaling" property:

$t \mapsto \lambda^z t\,, \quad \vec{x} \mapsto \lambda \vec{x}$, where $z \neq 1$ is called the "dynamical exponent",

instead of the more familiar conformal scalings:

$$t \mapsto \lambda t\,, \quad \vec{x} \mapsto \lambda \vec{x}\,.$$

• plus the following usual additional symmetries:
spatial translations + temporal translations + spatial rotations + parity (P) symmetry + time reversal (T) symmetry

• distinguish one spatial coordinate, call it r ($0 \leq r < \infty$), and let the dynamical scaling transformations act as

$$t \mapsto \lambda^z t \, , \ \ \vec{x} \mapsto \lambda \vec{x} \, , \ \ r \mapsto r/\lambda \, ,$$

where $z \neq 1$ again and $\vec{x}$ denotes a $(D-2)$-dimensional vector.
• Then one ends up with the D-dimensional static Lifshitz spacetime suitable for AdS/CFT games:

$$ds^2 = -\frac{r^{2z}}{\ell^{2z}} \, dt^2 + \frac{\ell^2}{r^2} \, dr^2 + \frac{r^2}{\ell^2} \left(\sum_{i=1}^{D-2} dx_i^2 \right) , \qquad (1)$$

• When $z = 1$, usual AdS$_D$ metric with $SO(D-1, 2)$ symmetry.
• the length scale set by $\ell > 0$.

A big digression: Conserved gravitational Killing charges

• a generic gravitational model derived from a diff. invariant action

$$\Phi_{ab} = \kappa \, \tau_{ab} \, , \tag{2}$$

τ_{ab}: the energy-momentum of a covariantly conserved source,
κ: basically a gravitational coupling constant.

• $\exists$ a background metric $\bar{g}_{ab}$ for which $\bar{\Phi}_{ab}(\bar{g}) = 0$ for $\tau_{ab} = 0$.

• linearize a generic metric g_{ab} which asymptotically approaches to the background $\bar{g}_{ab}$ in the usual way:

$$g_{ab} = \bar{g}_{ab} + h_{ab} \, ,$$

where the deviation h_{ab} should vanish "sufficiently rapidly" as one approaches the "background at infinity".

• linearize the full field equations (2) about the background:

$$\Phi^{L}_{ab} = \kappa\, T_{ab}\,, \tag{3}$$

where Φ^{L}_{ab} is linear in h_{ab} and the RHS of (3) contains all the remaining nonlinear parts as well as τ_{ab}.

• $\bar{g}_{ab}$: responsible for raising and lowering indices and defines the covariant derivative $\bar{\nabla}_a$.

• $\exists$ at least one globally defined background Killing vector $\bar{\xi}^a$

$$\bar{\nabla}_a \bar{\xi}_b + \bar{\nabla}_b \bar{\xi}_a = 0\,. \tag{4}$$

• Then the Bianchi identity of the full theory $\Longrightarrow\ \bar{\nabla}_a T^{ab} = 0$. Together with (4), one arrives at

$$\bar{\nabla}_a(T^{ab}\,\bar{\xi}_b) = 0 = \partial_a(\sqrt{-\bar{g}}\;T^{ab}\,\bar{\xi}_b)\,, \tag{5}$$

which can be used to construct a conserved Killing charge provided $T^{ab}\,\bar{\xi}_b = \bar{\nabla}_b\mathcal{F}^{ab}$ for an antisymmetric tensor $\mathcal{F}^{ab}$ modulo terms that vanish on-shell.

• Then the conserved Killing charge reads

$$Q^a(\bar{\xi}) = \int_\Sigma d^{D-1}x\,\sqrt{-\bar{g}}\,\Phi_L^{ab}\,\bar{\xi}_b = \int_{\partial\Sigma} d\bar{S}_b\mathcal{F}^{ab}\,, \tag{6}$$

Σ: the $(D-1)$-dim. hypersurface of the D-dim. spacetime,
$\partial\Sigma$: its $(D-2)$-dim. boundary with the surface element $d\bar{S}$.

The upshot:

• The procedure described above has been applied to a generic quadratic curvature gravity theory described by the action

$$I = \int d^D x \sqrt{-g} \Big(\frac{1}{\kappa}(R + 2\Lambda_0) + \alpha R^2 + \beta R_{ab} R^{ab} + \gamma (R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2) \Big),$$

Λ_0: the bare cosmological constant,
κ: related to the D-dim. Newton's constant G_D.

• Q^a, i.e. $\mathcal{F}^{ab}$, for a generic background (satisfying certain requirements) has been calculated. Let me just flash the answer:

$$\Phi_L^{ab}\bar{\xi}_b = \bar{\nabla}_b \mathcal{F}^{ab} + h^{ab}\bar{\Phi}_{bc}\bar{\xi}^c + \frac{1}{2}\bar{\xi}^a\bar{\Phi}_{bc}h^{bc} - \frac{1}{2}h\bar{\Phi}^{ab}\bar{\xi}_b \,, \qquad (7)$$

where

$$\mathcal{F}^{ab} = -\mathcal{F}^{ba} = \frac{1}{\kappa}\mathcal{F}_E^{ab} + \alpha\mathcal{F}_\alpha^{ab} + \beta\mathcal{F}_\beta^{ab} + \gamma\mathcal{F}_\gamma^{ab} \,,$$

$$\begin{aligned}
\mathcal{F}_E^{ab} &\equiv \bar{\xi}_c\bar{\nabla}^{[a}h^{b]c} + \bar{\xi}^{[b}\bar{\nabla}_c h^{a]c} + h^{c[b}\bar{\nabla}_c\bar{\xi}^{a]} + \bar{\xi}^{[a}\bar{\nabla}^{b]}h + \frac{1}{2}h\bar{\nabla}^{[a}\bar{\xi}^{b]}, \\
\mathcal{F}_\alpha^{ab} &\equiv 2\bar{R}\mathcal{F}_E^{ab} + 2\bar{\xi}^{[b}h^{a]c}\bar{\nabla}_c\bar{R} + 4\bar{\xi}^{[a}\bar{\nabla}^{b]}R_L + 2R_L\bar{\nabla}^{[a}\bar{\xi}^{b]}, \\
\mathcal{F}_\beta^{ab} &\equiv \bar{\xi}^{[a}\bar{\nabla}^{b]}R_L + 2\bar{\xi}^c\bar{\nabla}^{[b}(R^{a]}{}_c)_L + 2(R^{[b}{}_c)_L\bar{\nabla}^{a]}\bar{\xi}^c \\
&\quad + h_{cd}\bar{\xi}^{[b}\bar{\nabla}^{a]}\bar{R}^{cd} + 2h^{c[a}\bar{\xi}_d\bar{\nabla}_c\bar{R}^{b]d} + 2\bar{R}^{c[a}\bar{\xi}_d\bar{\nabla}_c h^{b]d} \\
&\quad + h\bar{\xi}_c\bar{\nabla}^{[b}\bar{R}^{a]c} + 2\bar{R}^{c[b}h^{a]d}\bar{\nabla}_c\bar{\xi}_d + h\bar{R}^{c[b}\bar{\nabla}^{a]}\bar{\xi}_c \\
&\quad + 2\bar{R}^{c[b}\bar{\xi}^d\bar{\nabla}_d h^{a]}{}_c + \bar{\xi}^{[a}\bar{R}^{b]c}\bar{\nabla}_c h + \bar{R}^{cd}\bar{\xi}^{[a}\bar{\nabla}^{b]}h_{cd} \\
&\quad + 2\bar{\xi}^d\bar{R}_{cd}\bar{\nabla}^{[b}h^{a]c} + 2\bar{R}^{c[a}\bar{\xi}^{b]}\bar{\nabla}^d h_{cd} + 2\bar{\xi}^d\bar{R}^{c[b}\bar{\nabla}^{a]}h_{cd},
\end{aligned}$$

$$
\begin{aligned}
\mathcal{F}_\gamma^{ab} \equiv\ & 2\bar{R}\mathcal{F}_E^{ab} + 2\bar{R}^{[ba]cd}\bar{\xi}_d\bar{\nabla}_c h + 4\bar{\xi}_c\bar{R}^{c[b}\bar{\nabla}^{a]}h + 4\bar{R}^{c[a}\bar{\xi}^{b]}\bar{\nabla}_c h \\
& + 2h\bar{R}^{c[ab]d}\bar{\nabla}_d\bar{\xi}_c + 4h\bar{R}^{c[a}\bar{\nabla}^{b]}\bar{\xi}_c + 4\bar{\xi}_d\bar{R}^{dec[a}\bar{\nabla}_c h^{b]}{}_e \\
& + 4\bar{\xi}_d\bar{R}^{dec[a}\bar{\nabla}_e h^{b]}{}_c + 4\bar{\xi}_d\bar{R}^{dec[b}\bar{\nabla}^{a]}h_{ce} + 4\bar{\xi}^{[a}\bar{R}^{b]cde}\bar{\nabla}_d h_{ce} \\
& + 2\bar{R}^{[ab]cd}\bar{\xi}^e\bar{\nabla}_c h_{de} + 2\bar{\xi}_d h_{ce}\bar{\nabla}^c\bar{R}^{[ab]de} + 4\bar{\xi}_d\bar{R}^{d[a}\bar{\nabla}_c h^{b]c} \\
& + 4h_{ce}\bar{R}^{dec[a}\bar{\nabla}^{b]}\bar{\xi}_d + 4\bar{R}^{[ab]cd}h_{ce}\bar{\nabla}_d\bar{\xi}^e + 4\bar{R}_{cd}\bar{\xi}^d\bar{\nabla}^{[b}h^{a]c} \\
& + 4\bar{\xi}^d\bar{R}^{c[a}\bar{\nabla}^{b]}h_{cd} + 4\bar{\xi}^d\bar{R}_c{}^{[b}\bar{\nabla}_d h^{a]c} + 4\bar{\xi}^d\bar{R}^{c[b}\bar{\nabla}_c h^{a]}{}_d \\
& + 8\bar{R}^{cd}\bar{\xi}^{[a}\bar{\nabla}_c h^{b]}{}_d + 4\bar{R}^{cd}\bar{\xi}^{[b}\bar{\nabla}^{a]}h_{cd} + 2\bar{R}^{cd}h_{cd}\bar{\nabla}^{[b}\bar{\xi}^{a]} \\
& + 4\bar{\xi}_d h^{c[a}\bar{\nabla}_c\bar{R}^{b]d} + 4h^{cd}\bar{\xi}^{[b}\bar{\nabla}_c\bar{R}^{a]}{}_d + 4h^{c[a}\bar{R}^{b]d}\bar{\nabla}_c\bar{\xi}_d \\
& + 8h_{cd}\bar{R}^{c[b}\bar{\nabla}^{a]}\bar{\xi}^d + 2\bar{\xi}^{[a}h^{b]c}\bar{\nabla}_c\bar{R},
\end{aligned}
$$

Checks:

• $Q^a(\bar{\xi})$ reduces to its correct counterpart when the background $\bar{g}_{ab}$ is taken to be a space of constant curvature.
• $Q^a(\bar{\xi})$ is indeed background gauge invariant, i.e. when the deviation h_{ab} transforms as

$$\delta_{\bar{\zeta}} h_{ab} = \bar{\nabla}_a \bar{\zeta}_b + \bar{\nabla}_b \bar{\zeta}_a \tag{8}$$

under an infinitesimal diffeomorphism generated by a vector $\bar{\zeta}^a$, $Q^a(\bar{\xi})$ is left invariant.

• Having developed all this "technology", apply the general charge formula $Q^a(\bar{\xi})$ to compute the energies of the static Lifshitz black holes presented by E. Ayon-Beato, *et al.* in JHEP **1004**, 030 (2010) [arXiv:1001.2361 [hep-th]], which are solutions of some quadratic curvature gravity theory.

• Static Lifshitz BH solutions are of the generic form

$$ds^2 = -\frac{r^{2z}}{\ell^{2z}} f(r) dt^2 + \frac{\ell^2}{r^2} \frac{dr^2}{f(r)} + \frac{r^2}{\ell^2} \Big(\sum_{i=1}^{D-2} dx_i^2 \Big), \tag{9}$$

where the function $f(r)$ and the range of the dynamical exponent z depend on the specific quadratic curvature gravity theory under consideration.

• Typically the function $f(r)$ has a behavior s.t. the boundary of the Lifshitz black hole is located at infinite r, and the background is the static Lifshitz spacetime, by taking $f(r) \to 1$ as $r \to \infty$.

• The background admits $\bar{\xi}^a = -(\partial/\partial t)^a$ as a timelike Killing vector, so in principle once can calculate the Killing energy of the static Lifshitz black holes, which reads

$$E = \lim_{r\to\infty} \int_{\partial\Sigma} d^{D-2}x \, \frac{r^{z+D-3}}{\ell^{z+D-3}} \, \mathcal{F}^{tr} = \Omega_{D-2} \left(\lim_{r\to\infty} \frac{r^{z+D-3}}{\ell^{z+D-3}} \, \mathcal{F}^{tr} \right).$$

for the case at hand.
Ω_{D-2}: the finite contribution of the $(D-2)$-dimensional integration over the ranges of the spatial coordinates x_i.

- Wald entropy gives

$$S = -2\pi\,\Omega_{D-2}\,\frac{r_+^{D-2}}{\ell^{D-2}}\left[\frac{\delta\mathcal{L}}{\delta R_{abcd}}\,\varepsilon_{ab}\,\varepsilon_{cd}\right]_{r=r_+},$$

$r = r_+$: the location of the event horizon, i.e. the largest positive real root of $f(r)$, i.e. $f(r_+) = 0$.
ε_{ab}: the binormal to the event horizon given by

$$\varepsilon_{ab} = \Big(\frac{r^{2z-1}}{\ell^{z-1}}\frac{(2zf(r) + rf'(r))}{r_+^{z+1}f'(r_+)}\Big)\delta_{ar}\delta_{bt}.$$

- Surface gravity $\varkappa$

$$\varkappa = \frac{1}{2}\frac{r_+^{z+1}}{\ell^{z+1}}f'(r_+)\,,$$

and the temperature is $T = \varkappa/(2\pi)$.

The energy and the entropy of Lifshitz black holes for $z > 2 - D$:

• Instead of boring you with the gory details of all the different special classes of BH solutions considered, let me just concentrate on one special class now, that demonstrates the basic features encountered.

• This class of solutions is in Sec 3.1 of E. Ayon-Beato, *et al.* and they have

$$f(r) = 1 - \frac{M\ell^{(z+D-2)/2}}{r^{(z+D-2)/2}},$$

M: a free parameter.

This is a solution to the generic quadratic curvature theory for $D \geq 5$ and describes a BH when $z > 2 - D$.

- Following the steps described before, one finds

$$E = \frac{M^2\,(z-1)\,p(z)}{\ell\,(z+D-2)\,q(z)}\,\Omega_{D-2}\,,$$

where

$$\begin{aligned} p(z) &= -8(D-2)\Big(9z^4-(D+5)z^3-(D-2)\times \\ &\quad \big[3(D-5)z^2+(D^2-5D+10)z-(D^2-4)\big]\Big)\,, \\ q(z) &= 27z^4-4(27D-45)z^3-(D-2)\times \\ &\quad \big[2(5D-116)z^2+4(D^2-D+30)z+(D+2)(D-2)^2\big]\,. \end{aligned}$$

- Likewise

$$\begin{aligned} T &= \frac{(z+D-2)}{8\pi\ell}\,M^{\frac{2z}{z+D-2}}>0 \quad \text{and} \\ S &= -4\pi\,\Omega_{D-2}\,\frac{(3z^2+(D^2-4))\,(D^2-(2-3z)^2)}{q(z)}\,M^{\frac{2(D-2)}{z+D-2}}\,. \end{aligned}$$

• The conformal limit $z = 1$ of this family is an asymptotically AdS BH solution with

$$\kappa = 1\,,\ \Lambda_0 = \frac{(D-1)(D-2)}{4\ell^2}\,,\ \gamma = \frac{\ell^2}{2(D-3)(D-4)}\,,\ \alpha = \beta = 0\,,$$

and has $E = 0$ for all $D \geq 5$, but

$$T = \frac{(D-1)}{8\pi\ell}\, M^{\frac{2}{D-1}} > 0 \quad \text{and} \quad S_W = 4\pi\,\Omega_{D-2}\, M^{\frac{2(D-2)}{D-1}} > 0\,.$$

• It is obvious from these, and also from the generic values of E, T and S, that the 1st law of BH thermodynamics in the form $dE = TdS_W$ does not hold for a generic member of this class of static Lifshitz BHs.

The possible reasons for the failure of $dE = TdS_W$:

• The generalized Killing charge definition... Results must be checked with other energy definitions (e.g. those that rely on 'holographic renormalization technology') and a comparison needed. (Ex: warped AdS_3 BH in NMG, Lifshitz BH of NMG)

• The Wald entropy... A careful scrutiny of S_W above (and other examples I do not consider here) indicates that the Wald entropy does not always satisfy the 2nd law of BH thermodynamics, i.e. that $S_W \geq 0$ does not always hold. Why? Horizon topology?

• For a generic quadratic curvature gravity theory in $D \geq 5$, there are also active massive tensor and/or massive scalar gravitons apart from the usual massless gravitons. Maybe these massive modes modify the 1st law of BH thermodynamics.

Stationary Lifshitz spacetimes:

- Replace the two separate "P symmetry" + "T symmetry" requirements with the weaker "PT symmetry" invariance, i.e. ask for the following symmetries:

dynamical scaling transformations + spatial translations + temporal translations + spatial rotations + PT symmetry

- Then one ends up with the D-dimensional stationary Lifshitz spacetime:

$$ds^2 = -\frac{r^{2z}}{\ell^{2z}}\,dt^2 + 2\omega\,\frac{r^{z+1}}{\ell^{z+1}}\,dt\,d\phi + \frac{r^2}{\ell^2}\,d\phi^2 + \frac{\ell^2}{r^2}\,dr^2 + \frac{r^2}{\ell^2}\left(\sum_{i=1}^{D-3} dx_i^2\right), \tag{10}$$

- For later convenience, $x_{D-2} \equiv \phi$ is distinguished from the remaining x_i $(1 \leq i \leq D-3)$.
- the static Lifshitz spacetime is obtained when ω, the dimensionless "rotation parameter", is set to 0.

R^2-corrected gravity theory:

- The action is simply

$$I = \int d^D x \sqrt{-g} \left(R + 2\Lambda + \alpha R^2 \right), \tag{11}$$

where Λ is the cosmological constant and α is a coupling constant .

The first result:

Stationary Lifshitz spacetime (10) solves the field equations following from the action (11) for generic values of the parameters z and ω in any $D \geq 3$, provided that α and Λ are tuned as

$$\alpha = \frac{1}{8\Lambda}, \tag{12}$$

$$\Lambda = \frac{2D^2 + 3(z-1)^2 + 2D(2z-3)}{8\ell^2} + \frac{(z-1)^2}{8\ell^2(1+\omega^2)}. \tag{13}$$

The main result:

Provided that α and Λ are precisely as in (12) and (13), the metric

$$
\begin{aligned}
ds^2 &= -\frac{r^{2z}}{\ell^{2z}}\, h(r)\, dt^2 + \frac{r^2}{\ell^2}\left(d\phi + \omega\, \frac{\ell^2}{r^2}\, dt\right)^2 \\
&\quad + \frac{\ell^2}{r^2}\, \frac{dr^2}{h(r)} + \frac{r^2}{\ell^2}\left(\sum_{i=1}^{D-3} dx_i^2\right), \text{ where} \qquad (14)\\
h(r) &\equiv c + k\, \frac{\ell^{2(1+z)}}{r^{2(1+z)}} + M^-\, \frac{\ell^{p_-}}{r^{p_-}} + M^+\, \frac{\ell^{p_+}}{r^{p_+}}, \text{ with} \qquad (15)\\
c &\equiv \frac{4\,\ell^2\,\Lambda}{2z^2 + (D-2)(2z+D-1)}, \qquad (16)\\
k &\equiv \frac{2\,\omega^2}{D^2 - 7D + 14 - 2z(D-3)}, \text{ and} \qquad (17)\\
p_\pm &\equiv \frac{1}{2}\left(3z + 2(D-2) \pm \sqrt{z^2 + 4(D-2)(z-1)}\right), \qquad (18)
\end{aligned}
$$

solves the field equations of the action (11) for any $D \geq 3$.

Remarks:

• Note that the coefficients c and k are completely determined by z and ω, whereas the integration constants $M^{\pm}$ are left as free parameters.

• (1) and the static version of the metric (14) [i.e. the one with $\omega = 0$ for which $c = 1$, $k = 0$, the relations (12) and (13) for α and Λ, and the metric function $h(r)$ in (15) are simplified accordingly] were first presented by E. Ayon-Beato, *et al.* in JHEP **1004**, 030 (2010) [arXiv:1001.2361 [hep-th]].

• However with ω on, (10) and the stationary metric (14) (with the accompanying equations (12), (13) and (15)-(18), respectively) are clearly more general.

• In the conformal limit $z = 1$ with $D = 3$, the metric (14) becomes identical to the BTZ metric when one sets $M^{+} = 0$, $M^{-} = -M < 0$ and $\omega = -j/2$.

• The curvature scalars of the metrics (10) and (14) are both given by $R = -4\Lambda$ precisely. This allows for the casting of the action (11) into the form

$$I = \frac{1}{8\Lambda} \int d^D x \sqrt{-g}\, (R + 4\Lambda)^2 \,.$$

In principle, this theory can be mapped into a scalar-tensor theory by a conformal transformation of the metric with conformal factor $\Omega^2 = 1 + 2\alpha R$. However, for the particular value of $\alpha = \Lambda/8$, this does not work since $R = -4\Lambda$ as well. So this is a genuine theory!

• To have $h(r)$ real, one needs

$$\begin{aligned} z < z_- &\equiv 4 - 2D - 2\sqrt{(D-1)(D-2)} < 0 \quad \text{or} \\ z > z_+ &\equiv 4 - 2D + 2\sqrt{(D-1)(D-2)} > 0\,. \end{aligned}$$

For the metric (14) to describe a black hole, a careful analysis further chooses the branch $z > z_+ > 0$.

• One can construct analogous solution(s) of the form (14) for the critical value(s) of the dynamical exponent $z = z_{\pm}$ with logarithmic $h(r)$ function(s).
• In fact the region $z \in (z_-, z_+)$ is not excluded either! However, let me skip the details of it here!
• A careful consideration shows that both the energy E and the entropy S, as well as the angular momentum J, vanish for this class of solutions: $E = S = J = 0$. Perhaps this is not so surprising after all, since the action $I = 0$ at the first place for these solutions! (Recall my previous remark following $R = -4\Lambda$.)

Conclusions and open problems:

• As put forward by E. Ayon-Beato, *et al.*, one may think of these solutions (or their Euclidean counterparts) as some kind of "gravitational instantons".

• Stability of these solutions has not been analyzed yet.

• The implications of these solutions on the CMT side has not been studied yet.