HOMEWORK IV

(Due March 30, 2006)

Maude Lebowski: What do you do for recreation?
The Dude: Oh, the usual. I bowl. Drive around. The occasional acid flashback. Solve some Quantum Field theory questions.
The Big Lebowski (1998)

Q1: Let us recapitulate the Lagrangian and the Hamiltonian formulation of classical mechanics. Given the following Lagrangian

$$L(q, \dot{q}) = \frac{1}{2} G_{ab}(\bar{q}) \dot{q}^a \dot{q}^b + h_a(\bar{q}) \dot{q}^a - V(\bar{q}),$$

where, $q^a$ are $N$ coordinates, $G_{ab}$ is a symmetric invertible $N \times N$ matrix.

a: Find the Hamiltonian $H(\bar{p}, \bar{q})$. [Please note that $H$ is a function of the coordinates and the momenta.]
b: Find the Euler–Lagrange equations.
c: Find the Hamilton’s equations.
d: Check that Hamilton’s equations reproduce Euler–Lagrange equations.

Q2: Higher derivative theories are now widely used. This exercise is intended to study non-relativistic limit of such theories.
a: Find the Euler–Lagrange (or Ostrogradski) equations for the following action

$$S = \int dt \ L(q, \dot{q}, \ddot{q}).$$

b Also, generalize your answer to the case for which the Lagrangian depends up to the $N^\text{th}$ $t$-derivative of the coordinates $q(t)$.

Q3: A nice application of ”Ostrogradski”
a: Consider the Pais-Uhlenbeck oscillator which is given by the following lagrangian ( Observe that this is not your grandmother’s good old harmonic oscillator’)

$$L = \frac{1}{2} \left\{ \ddot{q}^2 - \left( \omega_1^2 + \omega_2^2 \right) q^2 + \omega_1^2 \omega_2^2 q^2 \right\} \tag{1}$$

where $\omega_1 \neq \omega_2$. Find the equations of motion ?
b: Find the classical solution?
c: Find the hamiltonian of the system?

**BONUS QUESTION** You will get extra ($\Gamma[\pi\sqrt{e}]$) points for this part.  

i): Find the canonical transformations that diagonalize the hamiltonian?

ii): What is the formula for the energy spectrum of the quantum version of this oscillator? (For this part, you need to get the previous part. If you cannot do the bonus part, don’t get upset!)