Measurement Process in Quantum Mechanics

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The representation of the states and the measurement process in quantum mechanics is very counterintuitive. This is mainly due to the difference between the treatments of classical and quantum mechanics. In addition to this, the source of the rules, in other words, the derivation of the rules from more intuitive ones is usually complicated and not treated in introductory textbooks. The purpose of this note is to state the mathematical representation of the measurement process in quantum mechanics in a clear way. There is, of course, no derivations, as these rules are not deduced from some other theory. For this reason, the basic rules are usually stated as postulates.

1 Representation of States and Observables

When physicists are analyzing a physical system, they use a mathematical representation of the system and its observable properties. For example, for a truck moving on a road, or a classical point particle moving in one-dimensional space, we describe the physical state of it by the pair of numbers \((x, v)\) where \(x\) is the position and \(v\) is the velocity\(^1\). For this situation, we say that \((x, v)\) denotes the state of the truck (or the particle) at a given time. By this we imply that we can calculate any property of the object (such as energy, acceleration, net force applied on it, power of energy given to it, etc.) if we know these two numbers\(^2\). In addition to these, once the value of the state \((x, v)\) is known at time \(t = 0\), we can compute all future states, i.e., the state at time \(t\). Hence, the pair of numbers \((x, v)\) contains all information about the system. In classical mechanics, therefore, the state of such an object is represented by the pair of numbers \((x, v)\).

In quantum mechanics, we also represent the state of a system by some mathematical structure. But, this time, the mathematics that we use is a little bit more complicated. For a particle in one dimension, instead of the pair of numbers \((x, v)\) that we use in classical mechanics, we need to use a wave-function \(\psi(x)\) in quantum mechanics. Instead of two real variables, we need to use infinitely many real variables to describe the same object. Somehow, the wavefunction \(\psi(x)\) contains all information that we can talk about the particle. It is therefore, the state for quantum mechanics.

\(^1\)In classical mechanics, we usually take this pair to be the position and momentum, \((x, p)\).

\(^2\)Of course, we are assuming that we know the forces that act on the object so that we can compute these forces as a function of the state.
For a particle in one dimension:
Classical Mechanics: state $\leftrightarrow (x,v)$
Quantum Mechanics: state $\leftrightarrow \psi = \psi(x)$

There is a similar situation for the observable properties of systems. For classical mechanics, any property of the system is mathematically represented, essentially by a function of the state, i.e., position and velocity. For example, for a particle in 1D, the energy can be given by the Hamiltonian function $E = H(p,x) = p^2/2m + V(x) = mv^2/2 + V(x)$. In quantum mechanics, observables are mathematically represented by linear operators. Obviously, computation of the value of an observable requires a separate treatment.

After this introduction, let us state the postulates pertaining to states and observables. Please note that these mathematical structures will be used to analyze/understand/compute the physical behavior of systems. First, the postulate for states.

For a physical system, any possible state of the system is represented by a wavefunction $\psi$.

Notes:

- The “wavefunction” $\psi$ is called by different names depending on the context, but they all mean the same thing. Possible names that you can meet are: wavefunction, state function, state vector, state ket, state.

- Although we call it a “function”, $\psi$ may assume the form of various mathematical structures depending on the system described. For example, $\psi$ might be a column matrix where all entries are numbers. It might be a column matrix where all entries are functions. For systems where particle numbers change, $\psi$ might have more complicated mathematical structure. The important point is, whatever the system is, however complicated it can be, there is always a suitable mathematical structure which we can use to discuss the quantum behavior of the system.

- For a single (spinless) particle, $\psi$ is usually considered as a function of position ($\psi(x)$ in one dimension, $\psi(x,y,z)$ in 3 dimensions). If there are two particles, then $\psi$ is a function of both positions (i.e., $\psi = \psi(x_1,y_1,z_1,x_2,y_2,z_2)$ where $(x_n,y_n,z_n)$ denotes the position of the $n$th particle ($n = 1,2$)). If there are $N$ particles, $\psi$ depends on the positions of all $N$ particles. If particles have spin or if particles can be created or destroyed, then mathematics become a bit more complicated.

- Just like the classical state depending on time$^3$, the quantum-mechanical state $\psi$ also depends on time. As a result, for a particle in 1D, we have $\psi = \psi(x,t)$. But, if we are talking about the state at a single time, we might suppress $t$ in order to simplify the notation. The time dependence of the state will be discussed elsewhere.

$^3$The classical state for a particle in 1D is $(x,v)$. As the particle moves, $x$ (and perhaps $v$) changes. Therefore, the state is time dependent.
It is possible to make a distinction between the “physical state” (an abstract concept) of a system and its representation $\psi$. But, most of the physicists do not make this distinction and directly call $\psi$ as the state.

Let us now talk about the representation of observables.

For any observable property $A$ of the system, there is a corresponding linear operator $\hat{A}$.

Notes:

- The word “observable” might confuse you, but it is a term that stands for any observable property. By observable, we mean properties that can be measured somehow by using some apparatus. You might think that there might be some “unobservable” properties of the systems as well. Probably there are, but there is no need to mention them here because they may confuse you. Typical observables are energy, position, momentum, some functions of these, etc.

- The linear operator $\hat{A}$ essentially acts on wavefunction-like mathematical objects and produces a mathematical object of the same type. In $\hat{A}\psi = \phi$, if $\psi$ is a function of position $x$, i.e., $\psi = \psi(x)$, then so is $\phi$.

- The operator $\hat{A}$ is called linear if it acts linearly: In other words, the relation

$$\hat{A}(c_1\psi_1 + c_2\psi_2) = c_1\hat{A}\psi_1 + c_2\hat{A}\psi_2$$

is satisfied for all complex numbers $c_1$, $c_2$ and “functions” $\psi_1$ and $\psi_2$. Apart from a few exceptions that we will not see, only linear operators are used in quantum mechanics. Quantum mechanics is a linear theory.

- Because of the property above, the wavefunctions $\psi$ are considered as elements of a vector space. In other words, for any two possible wavefunctions $\psi_1$ and $\psi_2$, we can form linear combinations to obtain other possible wavefunctions $\psi_3 = c_1\psi_1 + c_2\psi_2$. A linear combination is a mathematical operation done with vectors. For this reason, the wavefunctions might also be called as vectors. Instead of linear combination, physicists usually prefer the term superposition.

- In various places, we will meet expressions like $\hat{A}\psi = \phi$ where the operator acts on a function and produces another function. In very different contexts we might meet such expressions. Depending of the context, $\psi$ may or may not correspond to a state (i.e., state of physical system); usually it will be a state. Also, the produced function $\phi$ may or may not correspond to a state; usually $\phi$ will not be a state.

- Some people make a distinction between an observable $A$ (an abstract concept) and the linear operator $\hat{A}$ that represents it. But, most people do not make this distinction and directly call the operator $\hat{A}$ as the observable. In these notes, we will do the same. As a result, when we say “measuring $A$” or “measuring $\hat{A}$”, we will mean the same thing.
2 States where observables are definite

In the above two definitions of state and observable we have expressed only the mathematical structure that they correspond. We have not talked about how these two are related. Consider a system in state $\psi$ and consider an observable $\hat{A}$. What is the value of the observable $\hat{A}$ when it is measured in $\psi$? In this section, we will learn how to find the answer to this question. But we will start with the simplest cases.

First, a definition: If the equation

$$\hat{A}\phi = \lambda\phi$$

is satisfied for a complex number $\lambda$ and nonzero function $\phi$, then we call $\lambda$ as an eigenvalue of the observable $\hat{A}$ and call $\phi$ as the eigenvector. Of course, different names might be given to the eigenvector: Eigenfunction, eigenstate, eigenket. The set of all eigenvalues $\lambda$ forms the spectrum of $\hat{A}$.

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Suppose that $\psi$ is an eigenvector of an observable $\hat{A}$ with eigenvalue $a$. In other words, the relation $\hat{A}\psi = a\psi$ is satisfied.

- We say that the observable $\hat{A}$ has the definite value $a$ in the state $\psi$.
- If the observable $\hat{A}$ is measured when the system was in state $\psi$, then
  - we get the value $a$ with certainty (probability=1), and
  - the system remains in the same state $\psi$.

Notes:

- Note that there is no mention of how the measurement is carried out. Unfortunately, at this level, it is not possible to discuss the actual experiment and its connection to the formalism discussed here. For this reason, keep in mind that what we are discussing is just the mathematical representation of the measurement process.

- The situation discussed in the box is actually a definition of the cases where an observable has a definite value. If the state $\psi$ is not an eigenvector of the observable $\hat{A}$, we say that the observable has indefinite values in that state. The indefiniteness concept will be analyzed later.

- The eigenvalues of the linear operator $\hat{A}$ are identified as the measured values. The quantization is a result of this, because some operators have only a discrete set of eigenvalues.

- Of course, for physical observables, the eigenvalue should be a real number. Therefore, for such observables, the whole eigenvalue spectrum must be completely on the real line. We will discuss this later.
Exercises:

For a particle in one dimension, the momentum operator $\hat{p}_x = \hat{p}$ is given by \(^4\)

$$\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

The eigenvalues of the momentum operator form a continuum. The eigenvalue is usually expressed as $\hbar k$ where $k$, the wavenumber, is any real number. Let us show the corresponding eigenfunction as $\phi_k$. Therefore, we have

$$\hat{p}\phi_k = \hbar k \phi_k$$

$$\phi_k(x) = e^{ikx}.$$  

This function is a (plane) wave having a single wavelength $\lambda = 2\pi/k$, which is basically de Broglie relation. Now, using these, the following examples can be understood.

ex If the particle is in state $\psi = 5e^{ikx}$, then the momentum has the definite value of $\hbar k$. If the momentum is measured, we definitely get $\hbar k$. The overall constant 5 has no effect on this statement because it cannot change the fact that $\psi$ is also an eigenvector of $\hat{p}$.

At this point, let us note the following result.

If $\phi$ is an eigenvector of $A$ with eigenvalue $\lambda$, then for any nonzero complex number $c$ the function $c\phi$ is also an eigenvector of $A$ with the same eigenvalue $\lambda$.

$$A\phi = \lambda \phi$$

$$\Rightarrow$$

$$A(c\phi) = \lambda (c\phi).$$

ex If the particle is in state $\psi = (3 + 2i)e^{ikx}$, then the momentum again has the definite value of $\hbar k$.

ex If the particle is in state $\psi = 72ie^{-2ikx}$, then the momentum has the definite value of $-2\hbar k$.

ex If the particle is in state $\psi = 4 - 3i$, then the momentum has the definite value of 0. Note that in this case the wavefunction is constant (it does not depend on position) and thus we have $\hat{p}\psi = 0$. You can read the last equation as $\hat{p}\psi = 0\psi$, which means that $\psi$ is an eigenvector and 0 is the eigenvalue.

Note also that this case is no different than the cases treated above because $\psi(x) = (4 - 3i)\phi_{k=0}(x)$.

ex Suppose that the particle is in state $\psi = e^{2ikx}$ and we measure the observable $\hat{B} = 5\hat{p}^3$. In that case, you can see that $\psi$ is also an eigenvector of $\hat{B}$ with eigenvalue $40\hbar^3k^3$. Therefore, we say that in state $\psi$ the observable $B$ has the definite value of $40\hbar^3k^3$. Therefore, when we measure $\hat{B}$, we will get $40\hbar^3k^3$ with certainty and the state will not change.

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\(^4\)When we are in one dimension we usually suppress the subscript in $\hat{p}_x$ to simplify the notation.
We frequently meet operators that are defined as functions of some other operators. Suppose that $\hat{B} = f(\hat{A})$ is one such relation. Here, the function $f$ might be a simple function like polynomials, or it might be more complicated like the exponential functions. A nice rule to remember here is that all eigenvectors of $\hat{A}$ are also eigenvectors of $\hat{B}$ and there is a very simple relation between eigenvalues:

$$\hat{A}\psi = \lambda\psi \implies f(\hat{A})\psi = f(\lambda)\psi .$$

Consider, as a simple example, the kinetic energy operator $\hat{K} = \hat{p}^2/2m$ in one-dimension. $\hat{K}$ is a function of $\hat{p}$ and therefore $\phi_k$ (the eigenfunctions of momentum) are also eigenfunctions of the kinetic energy. The eigenvalue of kinetic energy operator corresponding to the eigenfunction $\phi_k$ is therefore $\hbar^2k^2/2m$.

**ex** The wavefunction $\psi = \phi_k(x) = e^{ikx}$ is an eigenfunction of kinetic energy $\hat{K}$ with eigenvalue $E = \hbar^2k^2/2m$. Therefore, the kinetic energy has the definite value of $E$ in this state.

**ex** The wavefunction $\tilde{\psi} = \phi_{-k}(x) = e^{-ikx}$ is also an eigenfunction of kinetic energy $\hat{K}$ with the same eigenvalue $E = \hbar^2k^2/2m$. Therefore, in this state too, kinetic energy has the definite of $E$.

**ex** In the above two examples, we have two distinct functions, namely $\phi_k$ and $\phi_{-k}$ which are eigenfunctions of the kinetic energy with the same eigenvalue $E$. In these case we say that the *eigenvalue E is degenerate*. The *degeneracy of E* is defined as the number of linearly independent eigenvectors that have this eigenvalue. In that case, the degeneracy is 2.

**ex** Note that arbitrary superpositions of these two functions will also be eigenfunctions of $\hat{K}$ with the same eigenvalue $E$. In other words,

$$\hat{K}(c_1\phi_k(x) + c_2\phi_{-k}(x)) = E(c_1\phi_k(x) + c_2\phi_{-k}(x)) \quad \left(\text{where } E = \frac{\hbar^2k^2}{2m}\right)$$

for all numbers $c_1$ and $c_2$. As a result of this, in all of the following states, the particle has a definite value $E$ of kinetic energy.

$$\psi_1 = \cos kx ,$$
$$\psi_2 = 2\sin kx ,$$
$$\psi_3 = 3\cos(kx + \alpha) ,$$
$$\psi_4 = 5\sin\left(kx + \frac{\pi}{6}\right) ,$$
$$\psi_5 = 2e^{ikx} + ie^{-ikx} .$$

In all of the examples above, the state is a superposition of the two functions $e^{\pm ikx}$.

**ex** Consider the wavefunction $\psi_1$ given above. The state can be expressed as the following superposition.

$$\psi_1 = \cos kx = \frac{1}{2}e^{ikx} + \frac{1}{2}e^{-ikx} .$$
In other words, $\psi_1$ is a superposition of two momentum eigenfunctions with momentum values $+\hbar k$ and $-\hbar k$. Therefore, the value of momentum is indefinite in the state $\psi_1$. On the other hand, the value of the kinetic energy $\hat{K} = \hat{p}^2 / 2m$ is definite in $\psi_1$. Because of the relation between $\hat{K}$ and $\hat{p}$, we can guess that, even though the momentum is indefinite, there are only two values that we can possibly talk about, namely $+\hbar k$ and $-\hbar k$. Any other value for momentum is inconsistent with the definiteness of kinetic energy at value $\hbar^2 k^2 / 2m$. We therefore conclude that, when particle is in state $\psi_1$, its momentum has two possible values of $\pm \hbar k$. This is also true for all the wavefunctions $\psi_2$ through $\psi_5$ written above. We will discuss the distribution of measurement results later.

Now, we can also pass to the case of a particle moving in 2D or 3D. In that case, all three components of the momentum are given by similar expressions. In other words

$$\hat{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial x}, \quad \hat{p}_y = \frac{\hbar}{i} \frac{\partial}{\partial y}, \quad \hat{p}_z = \frac{\hbar}{i} \frac{\partial}{\partial z}.$$ 

We can give the following examples which are extensions of the previous results.

**ex** In the state

$$\psi = e^{-ikx} y^3 e^{-(y^2+3z^2)/\alpha^2},$$

the momentum component $\hat{p}_x$ has the definite value of $-\hbar k$. But, the components $\hat{p}_y$ and $\hat{p}_z$ are indefinite. The kinetic energy is also indefinite.

**ex** In the state

$$\psi = e^{-ikx+3iky} \cos(5kz),$$

$\hat{p}_x$ has the definite value of $-\hbar k$, $\hat{p}_y$ has the definite value of $3\hbar k$, but, $\hat{p}_z$ is indefinite. The kinetic energy

$$\hat{K} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m}$$

is also definite with value $E = 35\hbar^2 k^2 / 2m$.

**ex** In the state

$$\psi = e^{3ikx-4ikz},$$

all three components of momentum have definite values: $\hat{p}_x$ has value $3\hbar k$, $\hat{p}_y$ has value $0$, and $\hat{p}_z$ has value $4\hbar k$. Since all components are definite, we can say that the momentum itself is definite. The momentum observable $\hat{p} = \hat{p}_x \mathbf{i} + \hat{p}_y \mathbf{j} + \hat{p}_z \mathbf{k}$ has the definite value of $(3\mathbf{i} - 4\mathbf{k})\hbar k$. The kinetic energy is also definite with value $E = 25\hbar^2 k^2 / 2m$.

Now consider the position observable $\hat{x}$. This operator is defined by the rule

$$\hat{x}\psi = \phi \quad \leftrightarrow \quad x\psi(x) = \phi(x).$$

Consider the eigenvalue equation for the position operator

$$\hat{x}\psi = a\psi$$
where the eigenvalue is denoted by $a$. It can be seen that the associated functional equation is

$$(x - a)\psi(x) = 0 .$$

Therefore, at points other than $a$, (namely for $x \neq a$) the function is zero. There can be only one point ($x = a$) where the function is nonzero. There is only one function that satisfies these rules: Dirac $\delta$ function.

$$\psi(x) = \delta(x - a) .$$

This is the position eigenstate, i.e., the state where position has the definite value of $a$.

**ex** In the state

$$\psi = e^{2ikx}\delta(y - b)$$

$\hat{p}_x$ is definite with value $2\hbar k$, $\hat{p}_y$ is indefinite and $\hat{p}_z$ is definite with value 0. Kinetic energy is indefinite. For positions, $y$ is definite at value $b$, but both $x$ and $z$ are indefinite. The radial distance observable $\hat{r} = \sqrt{\hat{x}^2 + \hat{y}^2 + \hat{z}^2}$ is also indefinite.

In this section, we have learned how to recognize the states where a given observable has a definite value. If the state is $\psi$, then an observable $\hat{A}$ has a definite value in this state if and only if $\psi$ is an eigenvector of $\hat{A}$. Also, the value of $\hat{A}$ is the same as the eigenvalue. This is one of many statements in quantum mechanics that have both a physical and a mathematical expression.

$$\begin{align*}
(\text{A has definite value} & \quad \text{in state } \psi) \quad \iff \quad (\text{\psi is an eigenvector of } \hat{A}) \\
(\text{A has definite value} & \quad \text{of } \lambda \text{ in state } \psi) \quad \iff \quad (\hat{A} \psi = \lambda \psi)
\end{align*}$$

It is important to recognize the physical expression by looking at the mathematical one. Of course, every physical statement should have a mathematical representation. Quantum mechanics is different from the classical mechanics in the respect that there is no intuitive connection between the physical and the mathematical statements. Forming such a connection is an ability you need to gain.

### 3 Indefiniteness

Let us now look at the cases where a given observable $\hat{A}$ is indefinite in a given state $\psi$. In such a case, $\psi$ is not an eigenstate of $\hat{A}$. The best we can do is to try to express $\psi$ as a superposition of states that have definite values of $A$ (i.e., eigenstates of $\hat{A}$). This turns out to be very useful. Let us state our first important rule.
Let $\phi_n$ denote the eigenvectors of observable $A$ and let $\lambda_n$ be the respective eigenvalues ($A\phi_n = \lambda_n\phi_n$). Suppose that the state $\psi$ can be expressed as a superposition of some of these eigenvectors,

$$\psi = c_1\phi_{n_1} + c_2\phi_{n_2} + \cdots + c_\ell\phi_{n_\ell},$$

where the coefficients $c_1, c_2, \ldots, c_\ell$ are non-zero complex numbers. If the observable $A$ is measured when the system is in state $\psi$, then

- we can obtain one of the values $\lambda_{n_1}$, $\lambda_{n_2}$, ..., $\lambda_{n_\ell}$ with some probabilities.
- No other value for $A$ can possibly be obtained.

Notes:

- The probabilities and how the outcome is determined is also important. But these are mathematically more involved and for this reason discussed below.

- The result is actually quite reasonable once we accept the rule about the definiteness of observables. If $\phi_{n_1}$ is a state where $A$ has definite value of $\lambda_{n_1}$ and $\phi_{n_2}$ is a state where $A$ has definite value of $\lambda_{n_2}$, then it is quite reasonable to assume that the superpositions of these two states will be associated with both $\lambda_{n_1}$ and $\lambda_{n_2}$.

- A natural question you may ask is what “indeterminateness” mean. You may understand this in the way that the postulate above is stated, i.e., if you measure this observable, the outcome can be any value among many possible values.

Can we define indefiniteness without using the word “measurement”? This is also possible, but probably more confusing. In the state

$$\psi = c_1\phi_{n_1} + c_2\phi_{n_2} + \cdots + c_\ell\phi_{n_\ell},$$

we may say that $A$ has all of the $\ell$ possible values of $\lambda_{n_1}$, $\lambda_{n_2}$, ..., and $\lambda_{n_\ell}$. This is a reasonable statement, but it is harder for us to understand.

What is certain is that, in the state $\psi$, it is not possible to have statements like “$A$ has this value” or “$A$ has that value”. A subsequent measurement of $A$ can always contradict with any of these statements.

- “Indefinite” and “uncertain” are used interchangeably.

Examples:

ex Consider the following state

$$\psi(x) = 2\delta(x - a) + i\delta(x - 3a)$$

for a particle in one-dimension. The state is a superposition of position eigenstates with position values $a$ and $3a$. Therefore
(1) In state $\psi$, particle is at two different places, $x = a$ and $x = 3a$, at the same time.

(2) If position of the particle is measured when it was in state $\psi$, we either find it to be at $x = a$ or find it to be at $x = 3a$.

We will use these two statements interchangeably. In statement (1), we speak about the position before it is measured. At this time, the position is still indefinite. Both $x = a$ and $x = 3a$ are still possible. For this reason, we may say that the particle is at both positions at the same time. Of course, this is a very non-classical statement. This is one of the places where quantum theory diverges from all classical theories.

In statement (2), we are talking about the situation after a measurement is carried out. In that case, position is measured. We do not have any indefiniteness at this point in time. The position is either $x = a$ or $x = 3a$.

Statements (1) and (2) state the same thing. But, one is expressed before the measurement where there is still indefiniteness and the other is stated after the measurement, where the position is now definite.

**ex** In the state

$$\psi = 3 \sin kx$$

for a particle in 1D, the momentum is indefinite. It is both $\hbar k$ or $-\hbar k$. In other words, the particle is both moving left and moving right, at the same time. If momentum is measured when the particle is in this state, we will find the momentum to be either $+\hbar k$ or $-\hbar k$.

We may also measure position in state $\psi$. To analyze the possible results, we need to expand $\psi$ in terms of position eigenstates, i.e., $\delta(x - a)$. But, this is simple

$$\psi(x) = \int_{-\infty}^{+\infty} 3 \sin ka \delta(x - a) da = \int_{-\infty}^{+\infty} c(a) \delta(x - a) da$$

Here, $c(a) = 3 \sin ka$ are the coefficients of the expansion. Obviously, $\psi$ is a superposition of position eigenstates $\delta(x - a)$ for those values of $a$ where $c(a) \neq 0$. Therefore, when position is measured, we can find the particle to be at infinitely many places except $x = 0, \pm \pi/k, \pm 2\pi/k, \ldots$.

**ex** In the state

$$\psi = i \cos kx$$

for a particle in 1D, the momentum is again indefinite with values $\pm \hbar k$. When momentum is measured, we find the momentum to be either $+\hbar k$ or $-\hbar k$. When position is measured, we will find the position to be anywhere except $x = \pm \pi/2k, \pm 3\pi/2k, \pm 5\pi/2k, \ldots$.

**ex** In the state

$$\psi = \cos^2 kx$$

for a particle in 1D, the momentum is indefinite with possible values being $0$, $+2\hbar k$ and $-2\hbar k$. Possible position values are same as above.
ex In the state

$$\psi = \cos^2 kx$$

for a particle in 1D, the momentum is indefinite with four possible values: $\pm \hbar k$ and $\pm 3\hbar k$. Possible position values are same as above.

ex For a particle moving in 3D, in the state

$$\psi = 5 \cos k(x + y)$$

we can state the following.

- The $x$-component of momentum $p_x$ is indefinite with two possible values $\pm \hbar k$.
- The $y$-component of momentum $p_y$ is also indefinite with two possible values $\pm \hbar k$.
- The $z$-component of momentum $p_z$ is definite with value 0.
- The kinetic energy is also definite with value $\hbar^2 k^2 / m$.

Now it is time to say something about the final state. Consider the state

$$\psi(x) = \cos kx = \frac{1}{2} e^{ikx} + \frac{1}{2} e^{-ikx}$$

and suppose that the momentum of this particle is measured. Before and after the measurement we have the following:

- Before the measurement, state is $\psi$ and therefore the momentum is indefinite. The particle's momentum is both $+\hbar k$ and $-\hbar k$ at the same time.
- After the measurement, the momentum is definite. Momentum has the definite value that is measured.

If we remember the rule that “definite value” is synonymous with “state being eigenvector” of the corresponding observable, we can see that the state after the measurement should change. Before the measurement, the momentum is indefinite and hence the state is not an eigenvector. This can be seen easily by $\hat{p}\psi = i\hbar k \sin kx$ which is not a constant multiple of $\psi$. After the measurement, the momentum is definite, and therefore the new state should be a momentum eigenstate. Depending on the outcome of the measurement, there are two possibilities.

If outcome is $+\hbar k$ then $\psi_{\text{after}} = ce^{ikx}$, if outcome is $-\hbar k$ then $\psi_{\text{after}} = c'e^{-ikx}$.

Here, $c$ and $c'$ are constants that we cannot determine right now. What is important is that, at the moment the measurement is carried, the state of the particle suddenly changes from $\psi$ to $\psi_{\text{after}}$. This is called the reduction or collapse of the wavefunction.
Let \( \phi_n \) denote the eigenvectors of observable \( \hat{A} \) and let \( \lambda_n \) be the respective eigenvalues (\( \hat{A}\phi_n = \lambda_n \phi_n \)). Suppose that the state \( \psi \) can be expressed as a superposition of eigenvectors with different eigenvalues

\[
\psi = c_1 \phi_{n_1} + c_2 \phi_{n_2} + \cdots + c_\ell \phi_{n_\ell} ,
\]

where \( c_1, c_2, \ldots, c_\ell \) are non-zero and \( \lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_\ell} \) are all distinct. If the observable \( A \) is measured when the system is in state \( \psi \), and the \( j \)th outcome \( \lambda_{n_j} \) is obtained, then the wavefunction suddenly changes (collapses) to

\[
\psi_{\text{after}} = c_n \phi_{n_j}
\]

after the measurement.

Notes:

- The constant in front should not confuse you. It has no effect on the physical state the wavefunction represents. We will compute it later by using some requirements on the wavefunction.

- The collapse of the wavefunction implies that the measurement process disturbs the state. During the measurement, we are inescapably interacting with the system and therefore we disturb it. Because of this interaction, we irreversibly change its state.

- The following aspect of the collapse is nice. If, right after the first measurement, we measure the same quantity for a second time, then we should obtain an identical outcome. Collapse ensures that this is so.

Examples:

ex A particle in 1D is in the state 
\[
\psi = 8 \cos^2 kx
\]

and momentum is measured. In this case there are three possible outcomes for the momentum, \(-2\hbar k\), 0 and \(+2\hbar k\). The expansion is

\[
\psi = 2e^{-2ikx} + 4 + 2e^{2ikx}
\]

and therefore the final state after the measurement can be tabulated as follows.

<table>
<thead>
<tr>
<th>outcome</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2\hbar k)</td>
<td>(e^{-2ikx})</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(+2\hbar k)</td>
<td>(e^{2ikx})</td>
</tr>
</tbody>
</table>

ex A particle in 1D is in the state 
\[
\psi = 8 \cos^2 kx
\]
and kinetic energy $\hat{K} = \hat{p}^2 / 2m$ is measured. In this case there are two possible outcomes: 0 and $E = 2\hbar^2 k^2 / m$. The associated expansion is

$$\psi = 4 + 2e^{-2ikx} + 2e^{2ikx}$$

where the first term corresponds to a state with energy 0 and the last two terms correspond to a state with energy $E = 2\hbar^2 k^2 / m$. Therefore the final state after the measurement can be tabulated as

<table>
<thead>
<tr>
<th>outcome</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E = 2\hbar^2 k^2 / m$</td>
<td>$e^{-2ikx} + e^{2ikx}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

It is important to note that there is no collapse to either $e^{-2ikx}$ or $e^{2ikx}$, even though both of these states are eigenstates of the kinetic energy as well. As the eigenvalue $E$ of $\hat{K}$ is degenerate, we can find infinitely many possible eigenfunctions of $\hat{K}$. The particle’s state will transform only to one of these eigenstates when the result $E$ is obtained. The exact collapsed state can be found by looking at the expansion of $\psi$ in eigenfunctions corresponding to distinct eigenvalues. In this particular case, the expansion of $\psi$ contains the particular superposition $e^{-2ikx} + e^{2ikx}$ and hence the collapse will be to this superposition.

**ex** A particle in 1D is in the state

$$\psi = \sin kx$$

and the kinetic energy is measured. In this case, we have

$$\psi = \frac{i}{2} e^{ikx} + \frac{i}{2} e^{-ikx}$$

but both of the terms are eigenstates of the kinetic energy with the same eigenvalue of $E = \hbar^2 k^2 / 2m$. In fact, $\psi$ is itself an eigenfunction of $\hat{K}$ with eigenvalue $E$. Therefore, the kinetic energy has a definite value in $\psi$. In such a case, we will get the result $E$ with certainty and there will be no collapse. In other words, the state after will continue to be the same state: $\psi_{\text{after}} = \psi = \sin kx$.

**ex** A particle in 3D is in the state

$$\psi = e^{ikx} + e^{ik(x+y)} + 2e^{iky}$$

and one of the components of momentum is measured. Depending on the component measured we have a different final state.

- If $p_x$ is measured, we can expand $\psi$ as the sum of two terms

$$\psi = (e^{ikx} + e^{ik(x+y)}) + (2e^{iky})$$

where the first term corresponds to a state where $p_x = \hbar k$ and the last term corresponds to a state where $p_x$ is 0. Therefore, depending on the outcome the final state will be
outcome | collapsed state
--------|------------------
\(\hbar k\) | \(e^{ikx} + e^{ik(x+y)}\)
0 | \(e^{iky}\)

- If \(p_y\) is measured, we can expand \(\psi\) as the sum of two terms

\[\psi = (e^{ikx}) + (e^{ik(x+y)} + 2e^{iky})\]

where the first term corresponds to a state where \(p_y\) is 0 and the last term corresponds to a state where \(p_y\) is \(\hbar k\). Therefore, depending on the outcome the final state will be

\[
\begin{array}{|c|c|}
\hline
\text{outcome} & \text{collapsed state} \\
\hline
\hbar k & e^{ik(x+y)} + 2e^{iky} \\
0 & e^{ikx} \\
\hline
\end{array}
\]

- If \(p_z\) is measured, we can see that \(\psi\) is itself an eigenstate of \(\hat{p}_z\) and therefore there will be no collapse

\[
\begin{array}{|c|c|}
\hline
\text{outcome} & \text{collapsed state} \\
\hline
0 & e^{ikx} + e^{ik(x+y)} + 2e^{iky} \\
\hline
\end{array}
\]

- If the kinetic energy \(\hat{K}\) is measured instead, then we can expand \(\psi\) as the sum of two terms

\[\psi = (e^{ik(x+y)}) + (e^{ikx} + 2e^{iky})\]

where the two terms are kinetic energy eigenstates. Therefore, the final state table is

\[
\begin{array}{|c|c|}
\hline
\text{outcome} & \text{collapsed state} \\
\hline
\hbar^2 k^2/2m & e^{ikx} + 2e^{iky} \\
\hbar^2 k^2/m & e^{ik(x+y)} \\
\hline
\end{array}
\]

- It is also possible to measure all three components of the momentum operator at the same time\(^5\). In other words, we can measure \(\hat{\mathbf{p}}\) directly. In this case, the “vector eigenvalues”, in other words, the measurement outcomes of all three terms of \(\psi\) are different. Hence, the table of outcomes and the collapsed states will be like

\[
\begin{array}{|c|c|}
\hline
\text{outcome for } \hat{\mathbf{p}} & \text{collapsed state} \\
\hline
\hbar k \mathbf{i} & e^{ikx} \\
\hbar k (\mathbf{i} + \mathbf{j}) & e^{ik(x+y)} \\
\hbar k \mathbf{j} & e^{iky} \\
\hline
\end{array}
\]

4 Probabilities

After the collapse is handled, we can now deal with the computation of the probabilities of different outcomes. It is obvious that the probabilities are somehow related with the expansion coefficients. It appears that the probabilities are proportional to the modulus squares of these coefficients. However, in order to be able to write down a general formula for the probabilities,

\(^5\) We will discuss the simultaneous measurement of two or more observables elsewhere.
it is necessary that the eigenstates of the observable in question are normalized in the same way.

We will say that a wavefunction $\psi$ is normalized if it satisfies the relation,
\[
\int |\psi|^2 = 1,
\]
where the integral is over the whole space. A related concept is the norm $\|\psi\|$ of a function $\psi$. It is defined by
\[
\|\psi\| = \sqrt{\int |\psi|^2}.
\]
Therefore, we say that $\psi$ is normalized when its norm is $\|\psi\| = 1$. The norm essentially corresponds to the length of a vector in a vector space. A normalized function, therefore, corresponds to a unit vector. If a function $\psi$ is not normalized ($\|\psi\| \neq 1$), then we can normalize it by dividing it with its norm:
\[
\psi_{\text{normalized}} = \frac{1}{\|\psi\|} \psi.
\]
In other words, if you divide a given vector by its length, you can obtain a unit vector!

If the state of a system is expanded in normalized eigenstates of an observable, then the probabilities of different outcomes are simply proportional to the modulus squares of the coefficients.

\[
\text{Let } \phi_n \text{ denote the normalized eigenvectors of observable } \hat{A} \text{ and let } \lambda_n \text{ be the respective eigenvalues } (\hat{A}\phi_n = \lambda_n \phi_n). \text{ Suppose that the state } \psi \text{ can be expressed as a superposition of eigenvectors with different eigenvalues}
\]
\[
\psi = c_1 \phi_{n_1} + c_2 \phi_{n_2} + \cdots + c_\ell \phi_{n_\ell},
\]
where $c_1, c_2, \ldots, c_\ell$ are non-zero and $\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_\ell}$ are all distinct.

If the observable $A$ is measured when the system is in state $\psi$, the $j$th outcome $\lambda_{n_j}$ is obtained with probability
\[
p_j = \frac{|c_j|^2}{|c_1|^2 + |c_2|^2 + \cdots + |c_\ell|^2}.
\]

Unfortunately, even this formula cannot be directly applied to the position and momentum measurements. Consider, for example, the momentum eigenstates, which are simply $\phi_k(x) = e^{ikx}$ in 1D. Therefore, their norm is
\[
\|\phi_k\|^2 = \int_{-\infty}^{+\infty} |\phi_k(x)|^2 dx = \int_{-\infty}^{+\infty} dx = \infty!
\]
Therefore, the norms of these functions are infinity! Multiplying these functions by a constant does not improve the situation. Similarly the position eigenstates: For \( \varphi_a(x) = \delta(x - a) \), we have

\[
\| \varphi_a \|^2 = \int_{-\infty}^{+\infty} \delta(x - a) \delta(x - a) \, dx = \delta(a - a) = \delta(0) = \infty!
\]

In other words, the position eigenstates are also unnormalizable.

However, it is possible to choose a uniform normalization for these eigenstates and just for the purposes of computing the probabilities this is enough. Basically, in \( \phi_k(x) = e^{ikx} \), all exponential functions are multiplied by 1 and therefore we can take this as an indication of “same normalization”. In short, if we have

\[
\psi(x) = c_1 e^{ik_1 x} + c_2 e^{ik_2 x} + \cdots + c_\ell e^{ik_\ell x}
\]

then the probability of finding the momentum to be \( \sim k_j \) is given by Eq. (2). Similarly for the position eigenstates. If the state is

\[
\psi(x) = c_1 \delta(x - a_1) + c_2 \delta(x - a_2) + \cdots + c_\ell \delta(x - a_\ell)
\]

then measuring position to be \( a_j \) is given by Eq. (2).

**Exercises:**

**ex** Let the system be in the state \( \psi = 5 \cos kx \) and the momentum is measured. In that case, we have

\[
\psi = \frac{5}{2} e^{ikx} + \frac{5}{2} e^{-ikx}
\]

and therefore the probabilities of obtaining the outcomes \( +hk \) and \( -hk \) are equal, namely \( 1/2 \). To simplify the explanation of the results, I will express the outcomes as a table.

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -hk )</td>
<td>1/2</td>
<td>( e^{-ikx} )</td>
</tr>
<tr>
<td>( +hk )</td>
<td>1/2</td>
<td>( e^{ikx} )</td>
</tr>
</tbody>
</table>

**ex** If the system is in the state \( \tilde{\psi} = \sin kx \) and the momentum is measured, in that case, we have

\[
\tilde{\psi} = \frac{1}{2i} e^{ikx} - \frac{1}{2i} e^{-ikx}
\]

and as a result, the outcomes of the experiment will be exactly same as the table given above. In other words, as long as the momentum \( p \) is measured, there is no difference between the state \( \psi \) above and the state \( \tilde{\psi} \).

But, please note that the states \( \psi \) and \( \tilde{\psi} \) are different states and therefore some physical properties of them are different. (For example, if position measurement is made: when the particles is in state \( \psi \), it can be found at the origin \( x = 0 \), but when it is in state \( \tilde{\psi} \), it is impossible to find the particle at \( x = 0 \).) However, as far as momentum measurement is concerned, these two states look identical.
ex Let the particle be in the state $\psi = \cos^2 kx$ and momentum is measured. In this case, the expansion is

$$\psi = \frac{1}{4} e^{-2ikx} + \frac{1}{2} + \frac{1}{4} e^{2ikx}$$

and therefore we have the following outcomes

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2\hbar k$</td>
<td>1/6</td>
<td>$e^{-2ikx}$</td>
</tr>
<tr>
<td>0</td>
<td>4/6</td>
<td>1</td>
</tr>
<tr>
<td>$+2\hbar k$</td>
<td>1/6</td>
<td>$e^{2ikx}$</td>
</tr>
</tbody>
</table>

ex Let the particle be in the state

$$\psi = 4 + 3i + 2ie^{ikx} - 3e^{-ikx}$$

and momentum is measured. In that case, the state is

$$\psi = c_1 + c_2 e^{ikx} + c_3 e^{-ikx}$$

and the sum of the squares of the moduli is $|c_1|^2 + |c_2|^2 + |c_3|^2 = 38$ and therefore we have

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>25/38</td>
<td>1</td>
</tr>
<tr>
<td>$-\hbar k$</td>
<td>4/38</td>
<td>$e^{-ikx}$</td>
</tr>
<tr>
<td>$+\hbar k$</td>
<td>9/38</td>
<td>$e^{ikx}$</td>
</tr>
</tbody>
</table>

ex Let the particle be in the state given in Eq. (3), but this time the kinetic energy is measured. In that case, there are two possible kinetic energy values. For the kinetic energy $\hbar^2 k^2/2m$, the collapse will end up with the coherent superposition $2ie^{ikx} - 3e^{-ikx}$ and we need to assign a probability value to this outcome. Of course, the total probability that “the momentum is either $+\hbar k$ or $-\hbar k$” is

$$\frac{|c_2|^2 + |c_3|^2}{|c_1|^2 + |c_2|^2 + |c_3|^2} = \frac{4}{38} + \frac{9}{38} = \frac{13}{38}.$$

Therefore, we will assign the same probability for the kinetic energy. In this case, our table becomes

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>25/38</td>
<td>1</td>
</tr>
<tr>
<td>$-\hbar k$</td>
<td>13/38</td>
<td>$2ie^{ikx} - 3e^{-ikx}$</td>
</tr>
</tbody>
</table>

ex We can now see a 3D example. Let the particle in 3D be in the state

$$\psi = (1 + 4i)e^{ikx} + (3i - 1)e^{ik(x+y)} + (2 - i)e^{iky}$$

and one of the components of momentum is measured. Depending on the component measured, the associated probabilities as follows:
– If $p_x$ is measured, the table is as follows

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>$hk$</td>
<td>$27/32$</td>
<td>$(1 + 4i)e^{ikx} + (3i - 1)e^{ik(y + a)}$</td>
</tr>
<tr>
<td>0</td>
<td>$5/32$</td>
<td>$e^{iky}$</td>
</tr>
</tbody>
</table>

– If $p_y$ is measured, the table is as follows

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>$hk$</td>
<td>$15/32$</td>
<td>$(3i - 1)e^{ik(x+y)} + (2 - i)e^{iky}$</td>
</tr>
<tr>
<td>0</td>
<td>$17/32$</td>
<td>$e^{ikx}$</td>
</tr>
</tbody>
</table>

– If $p_z$ is measured, we have a definite value and therefore the table is as follows

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$(1 + 4i)e^{ikx} + (3i - 1)e^{ik(x+y)} + (2 - i)e^{iky}$ (namely $\psi$)</td>
</tr>
</tbody>
</table>

– If all three components of momentum is measured at the same time, then the collapse will end up in the three terms that appear in $\psi$.

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>$hk\imath$</td>
<td>$17/32$</td>
<td>$e^{ikx}$</td>
</tr>
<tr>
<td>$hk(1 + \jmath)$</td>
<td>$10/32$</td>
<td>$e^{ik(x+y)}$</td>
</tr>
<tr>
<td>$hk\jmath$</td>
<td>$5/32$</td>
<td>$e^{iky}$</td>
</tr>
</tbody>
</table>

– If the kinetic energy $\hat{K}$ is measured in state $\psi$, then the table will be as follows

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h^2k^2/[2m]$</td>
<td>$22/32$</td>
<td>$(1 + 4i)e^{ikx} + (2 - i)e^{iky}$</td>
</tr>
<tr>
<td>$h^2k^2/[m]$</td>
<td>$10/32$</td>
<td>$e^{ik(x+y)}$</td>
</tr>
</tbody>
</table>

**ex** If the state is

$$\psi = e^{ikx}(\delta(y - a) + 2\delta(y)) + 3i\delta(y + a)$$

and the $y$-component of position is measured, then the table of outcomes is as follows.

<table>
<thead>
<tr>
<th>outcome for $y$</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-a$</td>
<td>$9/14$</td>
<td>$\delta(y + a)$</td>
</tr>
<tr>
<td>0</td>
<td>$4/14$</td>
<td>$e^{ikx}\delta(y)$</td>
</tr>
<tr>
<td>$a$</td>
<td>$1/14$</td>
<td>$e^{ik}x\delta(y - a)$</td>
</tr>
</tbody>
</table>

If $p_x$ is measured instead, then we have the table

<table>
<thead>
<tr>
<th>outcome for $p_x$</th>
<th>prob.</th>
<th>collapsed state</th>
</tr>
</thead>
<tbody>
<tr>
<td>$hk$</td>
<td>$5/14$</td>
<td>$e^{ikx}(\delta(y - a) + 2\delta(y))$</td>
</tr>
<tr>
<td>0</td>
<td>$9/14$</td>
<td>$\delta(y + a)$</td>
</tr>
</tbody>
</table>
Continuous Distributions

What happens if the state $\psi$ cannot be expressed as a superposition of a discrete number of plane waves or Dirac functions? For example, what happens if $\psi(x) = \exp(-x^2/4\sigma^2)$? For these, the formalism that we have explained above can be directly extended to the current situation. For example, we can write down the following position-eigenstate expansion:

$$\psi(x) = \int \psi(a) \delta(x - a) da = \int_{-\infty}^{+\infty} e^{-\frac{a^2}{4\sigma^2}} \delta(x - a) da .$$

This is an expansion of $\psi$ in terms of the position eigenstates $\delta(x - a)$. It is quite similar to the discrete expansions like

$$\tilde{\psi}(x) = \sum_i c_i \delta(x - a_i) .$$

The only difference between them is that one is a summation over a discrete index and the other is an integral over a continuous index. Ignoring that detail, we can see that

$$\left(\text{Probability of finding position to be in the interval } [a', a'']\right) = \frac{\int_{a'}^{a''} |\psi(a)|^2 da}{\int_{-\infty}^{+\infty} |\psi(a)|^2 da} .$$

In applications, we usually find it convenient to normalize the wavefunction so that

$$\int_{-\infty}^{+\infty} |\psi(a)|^2 da = 1 .$$

In such a case, we can simply express the probability as

$$\left(\text{Probability of finding position to be in the interval } [a', a'']\right) = \int_{a'}^{a''} |\psi(a)|^2 da .$$

Therefore, $|\psi(a)|^2$ is the probability density for position to be found around $x = a$.

A similar approach must be followed for the momentum distribution. In that case, we have to expand the wavefunction $\psi(x)$ in terms of the plane wave states $e^{ikx}$. The expansion can be expressed as follows:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx} dk .$$

Here, $\sqrt{2\pi}$ factor is used for convenience (it will be very convenient in Parseval’s theorem). Apparently, the expansion coefficients $\phi(k)$ is the Fourier transform of $\psi(x)$. In quantum mechanics we call $\phi(k)$ as “the momentum-space wavefunction”. The probability for momentum distribution can now be expressed as

$$\left(\text{Probability of finding momentum to be in the interval } [hk', hk'']\right) = \frac{\int_{hk'}^{hk''} |\phi(k)|^2 dk}{\int_{-\infty}^{+\infty} |\phi(k)|^2 dk} .$$

For a normalized wavefunction, Parseval’s theorem dictates that

$$\int_{-\infty}^{+\infty} |\phi(k)|^2 dk = \int_{-\infty}^{+\infty} |\psi(a)|^2 da = 1$$
and therefore the probability formula is simply

\[
\left( \text{Probability of finding momentum to be in the interval } [\hbar k', \hbar k''] \right) = \int_{k'}^{k''} |\phi(k)|^2 \, dk.
\]

Therefore, we can call \(|\phi(k)|^2\) as the probability density for momentum distribution.