HOMEWORK II
(Due March 30, 2009)

"In mathematics you don’t understand things. You just get used to them” – Johann von Neumann

Q1: Using the Fourier transform technique, solve the following equation

\[ \frac{d\vec{v}}{dt} + \gamma \vec{v} = -e \bar{E}(t) \]

where \( \bar{E}(t) = \bar{E}_0 e^{-\alpha t} \) for \( t > 0 \) and \( \vec{v}(0) = 0 \). Here \( \alpha > 0 \). [ What happens for the special case \( \gamma = \alpha \)?

Q2: Use Laplace transform to solve the following coupled differential equations

\[ x'(t) + y'(t) - 4y(t) = 1, \quad y'(t) + x(t) - 3y(t) = t^2, \] (1)

with the initial conditions \( x(0) = y(0) = 0 \)

Q3: a) Let \( J_n(t) \) be the Bessel functions, Find \( \mathcal{L}\{J_n(at)\} \), where \( a \) is a constant.

b) Show that

\[ \int_0^\infty dk e^{-kz} k J_1(ka) = \frac{a}{(z^2 + a^2)^{3/2}}, \quad \mathcal{R}(z) \geq 0. \] (2)

Q4: An undamped harmonic oscillator is driven with a force \( F(t) = F_0 \sin(wt) \), use the convolution theorem in Laplace transform method to find the position \( x(t) \), with the initial conditions \( x(0) = x'(0) = 0 \)

Q5: Show that

\[ \mathcal{L}^{-1}\left\{ \frac{1}{\sqrt{s^2 + 1}} \right\} = J_0(t) \] (3)

Q6: Using a 3-dimensional Fourier transform, solve the following equation

\[ -\nabla^2 \varphi(\vec{r}) + m^2 \varphi(\vec{r}) = q\delta(\vec{r}) \]
Q7: a) Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left\{ \frac{k^2}{(s^2 + k^2)^2} \right\}$$

b) Find the Laplace transform $\mathcal{L}\{L_n(at)\}$, if the $L_n(t)$ satisfies the following equation

$$tL_n''(t) + (1 - t)L_n'(t) + nL_n(t) = 0$$

You need to be careful with the boundary conditions. I will give you these $L_n(0) = L_n'(0) = 0$.

Q8: a) Find the Fourier sine transform of

$$f(t) = e^{-at}$$

b) Using the result in the previous part, evaluate

$$I = \int_0^\infty dw \frac{w \sin(w)}{(w^2 + a^2)^2}$$

Q9: Show that

$$\mathcal{L}^{-1}\left\{ \frac{\ln s}{s} \right\} = -\ln t - \gamma$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant.