6:13 Fall 2004 M327
HW #1 Solutions

Q1: \( ds^2 = a^2(\sin^2 \theta + \cos^2 \theta) d\phi^2 \)

a) at \( \phi = 0 \), we have an ellipse whose circumference can be approximated to be \( \pi (R_p + R_e) \) where \( R_p \) and \( R_e \) are Polar and Equatorial radii.
So we have
\[
\int_{0}^{10^6} ds = 2\pi a = \pi (R_p + R_e) \Rightarrow a = 6367.5 \text{ km}
\]

at \( \theta = \pi/2 \), we have a circle, therefore
\[
\int ds = \int_{0}^{\pi/2} a = R_e \Rightarrow a = \frac{R_e - R_p}{R_e + R_p} = 0.00165
\]

b) at some \( \theta \), we have
\[
d s = a \sin \theta (1 + \epsilon \sin^2 \theta) d\phi \Rightarrow c = \int ds = 2\pi a \sin \theta (1 + \epsilon \sin^2 \theta)
\]

\[
A = 4\pi a^2 \left( 1 + \frac{\epsilon}{3} \right)
\]

So due to \( \epsilon \) we have \( E \approx \frac{8\pi a^2}{3} \approx 560 \text{ km}^2 \)

[ Area of Turkey is \( \approx 844,578 \text{ km}^2 \)]

\[
Q2: \quad ds^2 = 9z_1 dx^2 + 29z_2 dx dy + 9z_2 dy^2
\]

a) let \( ds^2 = \sqrt{9z_1} dx^2 + \sqrt{9z_2} dy^2 \)

What is the angle btw \( \sqrt{9z_1} dx \) and \( \sqrt{9z_2} dy \)?

\[
\frac{\int dx dy}{\sqrt{9z_1} \sqrt{9z_2}} \Rightarrow \sin \theta = \frac{\int dx dy}{\sqrt{9z_1} \sqrt{9z_2}}
\]

\[
A = \int_{0}^{\pi} \sqrt{9z_1} \sqrt{9z_2} \sin \theta \ dx \ dy
\]

\[
\Rightarrow dA = \int_{\int} \sqrt{9z_1} \sqrt{9z_2} \ sin \theta \ dx \ dy
\]
\[ b) \quad g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad \sqrt{\det g} = \sqrt{g_{11} g_{22} - g_{12} g_{21}} \quad g_{ij} = g_{ij} \]

\[ ds^2 = \sqrt{\det g} \, dx^2 + dv^2 \]

\[ (1) \quad g_{11} \left( \frac{\partial x}{\partial u} \right)^2 + 2 g_{12} \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + g_{22} \left( \frac{\partial x}{\partial v} \right)^2 = \Omega^2 \]

\[ (2) \quad g_{11} \left( \frac{\partial y}{\partial u} \right)^2 + 2 g_{12} \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + g_{22} \left( \frac{\partial y}{\partial v} \right)^2 = \Omega^2 \]

\[ (3) \quad g_{11} \left( \frac{\partial z}{\partial u} \right)^2 + 2 g_{12} \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} + g_{22} \left( \frac{\partial z}{\partial v} \right)^2 = 0 \]

b) New variable \( \theta = \frac{\partial z}{\partial u} \), \( \psi = \psi \)

\[ d\theta = \frac{\partial z}{\partial u} \, du \quad dv = dv \]

\[ ds^2 = a^2 \left( \frac{d\theta}{du} \right)^2 + a^2 \sin^2 \theta \, (du)^2 \]

\[ \tan \left( \frac{\theta}{2} \right) = e^{w/2} \quad (c_2 \text{ is some constant which could be set to unity}) \]

Then \( \sin \theta = \frac{2}{\sqrt{1 + e^{2u} + e^{-2u}}} \)

So we have \[ ds^2 = \frac{4a^2}{(2 + e^{2u} + e^{-2u})} \left( (du)^2 + (dv)^2 \right) \]
Girards theorem: The sum of the internal angles of a "splendid triangle"—that is, a triangle whose sides are pieces of great circles—is given by

$$\sum \alpha_i = \pi + \text{Area}.$$  

Next value (560°) or 377° corresponds to a circle.

A-B-C is a triangle whose area is

$$\frac{1}{2} \text{Area}^2.$$