Notes: These are some problems that you may want to solve by yourself. Some of these problems might be solved in recitation hours.

Commutators

1. Consider the orbital angular momentum operator \( \vec{L} = \vec{r} \times \vec{p} \) of a single particle. Compute the following commutators by using the fundamental position-momentum commutators \( [x_i, p_j] = i\hbar \delta_{ij} \).

   (a) \( [L_z, L_x] \)
   (b) \( [L_z, x] \)
   (c) \( [L_z, p_x] \)
   (d) \( [L_z, r^2] = [L_z, x^2 + y^2 + z^2] \)
   (e) \( [L_z, p^2] = [L_z, p_x^2 + p_y^2 + p_z^2] \)

   Note: The following commutator identity (the “derivative rule”) is usually useful.

\[
[A, P Q \cdots Z] = [A, P] Q \cdots Z + [A, Q] \cdots P Q \cdots Z + \cdots + P Q \cdots [A, Z] \\
[P Q \cdots Z, A] = [P, A] Q \cdots Z + [P, Q] \cdots Z + \cdots + P Q \cdots [Z, A]
\]

2. Let \( \vec{L} \) be the total angular momentum of two particles, i.e., \( \vec{L} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 \). Convince yourself that \( \vec{L} \) satisfies the same commutation relations:

\[ [L_x, L_y] = i\hbar L_z , \quad \text{etc.} \]

3. Compute the following commutation relations by using only the fundamental angular momentum commutation relations \( [J_i, J_j] = i\hbar \sum_k \epsilon_{ijk} J_k \). (Here \( J^2 = \vec{J} \cdot \vec{J} = J_x^2 + J_y^2 + J_z^2 \) and \( J_\pm = J_x \pm iJ_y \).

   (a) \( [J_z, J^2] \)
   (b) \( [J_z, J_\pm] \)
   (c) \( [J_+, J_-] \)
   (d) \( [J_x, J^2] \)

4. An operator \( \vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k} \) is called a “vector operator” if their commutator with the angular momentum operators satisfy the relations

\[ [J_i, V_j] = i\hbar \sum_k \epsilon_{ijk} V_k , \quad \text{or equivalently} \quad [V_i, J_j] = i\hbar \sum_k \epsilon_{ijk} V_k . \]

In other words, we have \( [J_z, V_y] = i\hbar V_z \) etc.

Use these commutation relations to solve the following problems:
(a) If \( \vec{V} \) is a vector operator show that
\[
[J_x, V^2] = [J_x, V_x^2 + V_y^2 + V_z^2] = 0.
\]

(b) If \( \vec{V} \) and \( \vec{W} \) are vector operators show that
\[
[J_x, \vec{V} \cdot \vec{W}] = [J_x, V_x W_x + V_y W_y + V_z W_z] = 0.
\]

(c) If \( \vec{V} \) and \( \vec{W} \) are vector operators show that
\[
[J_x, (\vec{V} \times \vec{W})_y] = i\hbar (\vec{V} \times \vec{W})_z.
\]

**Note:** The physical meaning of these relations are as follows:

- If \( \vec{V} \) is a vector, then, when you physically rotate a system, the expectation values \( \langle \vec{V} \rangle \) also rotate in the way you expected, i.e., as vectors. Somehow, this fact is directly related with the commutation relations \( [J_i, V_j] = i\hbar \sum_k \epsilon_{ijk} V_k \).

- If \( A \) is an operator that does not change under rotations, (in other words, \( \langle A \rangle \) remains invariant when system is rotated), then this fact is somehow related with the commutation relations \( [J_i, A] = 0 \).

- The exercises in this problem just demonstrate that if \( \vec{V} \) and \( \vec{W} \) are vectors then \( \vec{V} \times \vec{W} \) is also a vector and both \( V^2 \) and \( \vec{V} \cdot \vec{W} \) are rotationally invariant.

5. The following question involves manipulation of arbitrary indices in symbolic form. You should be able to handle these kind of expressions as well.

(a) Let \( \epsilon_{ijk} \) be the Levi-Civita tensor. Convince yourself that the following equation is correct.
\[
\sum_k \epsilon_{ijk} \epsilon_{nmk} = \delta_{in} \delta_{jm} - \delta_{im} \delta_{jn}.
\]

(b) Use the identity above to show that
\[
A_i B_j - A_j B_i = \sum_k \epsilon_{ijk} (\vec{A} \times \vec{B})_k.
\]

(c) The components of the orbital angular momentum of a single particle are \( L_i = \sum_{jk} \epsilon_{ijk} x_j p_k \). Show that
\[
[L_i, L_j] = i\hbar (x_j p_i - x_i p_j)
\]
and then use the result in part (b) to obtain the angular momentum commutation relations.

(d) Obtain the following relations
\[
[L_i, x_j] = i\hbar \sum_k \epsilon_{ijk} x_k,
\]
\[
[L_i, p_j] = i\hbar \sum_k \epsilon_{ijk} p_k.
\]

*By the interpretation of problem 4, these relations show that \( \vec{r} \) and \( \vec{p} \) are indeed vector operators, just as we expected!*

(e) Show that, if \( [J_i, V_j] = i\hbar \sum_k \epsilon_{ijk} V_k \) then \( [J_i, V^2] = 0 \). (Here, \( V^2 = \sum_i V_i^2 \).)