PHYS 431
Problem Set II

Notes: Some more problems to be solved.

Angular Momentum Eigenstates

1. Compute the following expectation values in state $|j, m\rangle$.
   (a) $\langle J_x J_y \rangle$. (Is the result you obtained real? Or is it complex? Is there a problem if it is complex? Is $J_x J_y$ a hermitian operator? Is this an observable?)
   (b) $\langle J_y J_x \rangle$. (The same questions above can be asked here too.)
   (c) What is the difference between the answers of (a) and (b), i.e., $\langle J_x J_y \rangle - \langle J_y J_x \rangle = ?$
   (d) What is the sum, i.e., $\langle J_x J_y + J_y J_x \rangle = ?$
   (e) Let $\phi$ be an arbitrary angle. Then $\hat{n} = \cos \phi \hat{i} + \sin \phi \hat{j}$ is an arbitrary unit vector on $xy$-plane. The component of $\vec{J}$ along $\hat{n}$ is then
   $$J_n = \hat{n} \cdot \vec{J} = \cos \phi J_x + \sin \phi J_y .$$
   Show that, if the value of $\langle J^2_n \rangle$ does not depend on $\phi$, then we must necessarily have $\langle J^2_x \rangle = \langle J^2_y \rangle$ and $\langle J_x J_y + J_y J_x \rangle = 0$.

2. Show that $\langle J^2_x J_y \rangle = 0$ in states $|j, m\rangle$.

3. Consider the state $|3, 2\rangle$ and compute the following
   (a) $\langle J_x \rangle$, $\langle J_y \rangle$ and $\langle J_z \rangle$. Also write $\langle \vec{J} \rangle$.
   (b) $\langle J^2_x \rangle$, $\langle J^2_y \rangle$ and $\langle J^2_z \rangle$. Also compute $\langle J^2_x + J^2_y + J^2_z \rangle$ and compute $\langle J^2 \rangle$.
   (c) $\Delta J_x$, $\Delta J_y$ and $\Delta J_z$. Are these consistent with the known commutators $[J_x, J_z]$ etc.?  
   (d) If $A = J^2$ is considered as an observable, what is $\Delta A$? If $B = J^2_x + J^2_y$, what is $\Delta B$?
   (e) What is $\langle J_x J_y \rangle$ and $\langle J_y J_x \rangle$? Also compute $\langle J_x J_y + J_y J_x \rangle$.

4. Consider the state
   $$|\psi\rangle = N\left(3 |3, 2\rangle + (1 + 2i) |3, 1\rangle \right) .$$
   (1)
   (a) Find a value for $N$ so that $|\psi\rangle$ is normalized.
   (b) Find $\langle J_+ \rangle$ and use this to find $\langle J_- \rangle$, $\langle J_x \rangle$ and $\langle J_y \rangle$.
   (c) Find $\langle J_z \rangle$.
   (d) What is $\langle \vec{J} \rangle$?
   (e) First compute $J_x |\psi\rangle$ and using the norm of the result find $\langle J^2_x \rangle$. 

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(e') This time, first compute \( J_x^2 |\psi\rangle \) and use this to find \( \langle J_x^2 \rangle \).

(f) Find the uncertainties \( \Delta J_x, \Delta J_y \) and \( \Delta J_z \). Also find \( \Delta A \) where \( A = J^2 \).

Rotation

5. Consider a system which is initially prepared to be in the state \( |\psi\rangle \) given in Eq. (1) in problem 4. Suppose that the system is rotated around the \( z \)-axis by \( \alpha = \pi/2 \) radians. As a result of this the state changes to the rotated state \( |\psi'\rangle = D(z, \alpha) |\psi\rangle \).

(a) What is \( |\psi'\rangle \)?

(b) Find the expectation value \( \langle \vec{J} \rangle' \) in state \( |\psi'\rangle \).

(c) Compare \( \langle \vec{J} \rangle' \) with \( \langle \vec{J} \rangle \) found in 4 (d). Is it rotated in the way that you expect?

6. Consider the state

\[
|\phi\rangle = \frac{|1,1\rangle + |1,-1\rangle}{\sqrt{2}}
\]

(a) By computing show that \( \langle \vec{J} \rangle = 0 \).

(b) Even though the average angular momentum is zero, this state is not rotationally symmetric. Specifically, if you rotate this system around the \( z \) axis by \( \alpha \) radians \((0 < \alpha < 2\pi)\), then you get a state different from \( |\phi\rangle \). In other words, show that

\[
|\phi_{\text{rotated}}\rangle = D(z, \alpha) |\phi\rangle \neq (\text{phase factor}) |\phi\rangle
\]

Measurement

7. Consider the state

\[
|\psi\rangle = N\left( (1 + 4i) |3,2\rangle + (2 + 3i) |2,2\rangle + (3 - i) |2,1\rangle \right)
\]

(a) Find a value for \( N \) so that \( |\psi\rangle \) is normalized.

(b) Suppose that \( J_z \) is measured. Construct a table that shows (i) the outcomes, (ii) their probabilities and (iii) the final collapsed state.

<table>
<thead>
<tr>
<th>Outcomes</th>
<th>Probabilities</th>
<th>Final state</th>
</tr>
</thead>
</table>

(b') Use the table above to compute \( \langle J_z \rangle, \langle J_z^2 \rangle, \langle J_z^3 \rangle \) and \( \Delta J_z \). (Use statistical arguments, not expectation values.)

(c) Suppose that \( J^2 \) is measured. Make a similar table showing the measurement outcomes, probabilities and the collapsed states.

(c') Use the table above to compute \( \langle J^2 \rangle \).

(d) Suppose that \( B = J_x^2 + J_y^2 \) is measured. Make a similar table showing the measurement outcomes, probabilities and the collapsed states.

(d') Use the table above to compute \( \langle J_x^2 + J_y^2 \rangle \).