Spin of spin 1/2 particles

1. Let \( \hat{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k} \) be a unit vector (i.e., \( n_x^2 + n_y^2 + n_z^2 = 1 \)). The component of spin along \( \hat{n} \) is defined as

\[
S_n = \hat{n} \cdot \vec{S} = n_x S_x + n_y S_y + n_z S_z = \frac{\hbar}{2} \sigma_n
\]

where

\[
\sigma_n = \hat{n} \cdot \vec{\sigma} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z = \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix}.
\]

Remember that \( S_n \) is a component of spin just like any other component, so it has the same physical properties. In this problem, you will verify this mathematically.

(a) Find the eigenvalues of \( \sigma_n \). Use this to find the eigenvalues of \( S_n \). Interpret this result in terms of isotropy of space.

(b) Use part (a) to find the eigenvalues of \( \sigma_n^2 \). Based on this, what do you think about what \( \sigma_n^2 \) is? To verify your guess, compute \( \sigma_n^2 \) explicitly by matrix product.

2. Let \( \alpha_n \) and \( \beta_n \) represent the normalized eigenvectors of \( \sigma_n \) corresponding to +1 and -1 respectively.

(a) Solve the eigenvalue problem for \( \sigma_x \), \( \sigma_y \) and \( \sigma_z \) to find the corresponding eigenvectors: \( \alpha_x, \ldots, \beta_z \).

(b) Check that

i. \{\( \alpha_x, \beta_x \)\} is an orthonormal basis of the Hilbert space of \( 2 \times 1 \) column vectors;
ii. \{\( \alpha_y, \beta_y \)\} is an orthonormal basis;
iii. \{\( \alpha_z, \beta_z \)\} is an orthonormal basis.

(c) Suppose that an electron spin is in the spin state given below

\[
\psi = N \begin{bmatrix} 2 + i \\ 2 \end{bmatrix}
\]

where \( N \) is a normalization constant. Find a value for \( N \) so that \( \psi \) is normalized.

(d) Suppose that \( S_z \) is measured when the electron is in state \( \psi \). (i) Which values can be obtained, (ii) what are the probabilities of obtaining each and (iii) what is the final collapsed state in each case?

(e) Suppose that \( S_x \) is measured when the electron is in state \( \psi \). Answer the questions that are asked in part (d).

(f) Suppose that \( S_y \) is measured when the electron is in state \( \psi \). Answer the questions that are asked in part (d).
Using the measurement outcomes and their probabilities you have obtained in parts (d), (e) and (f), find the statistical averages $\langle S_x \rangle$, $\langle S_y \rangle$ and $\langle S_z \rangle$. What is $\langle \vec{S} \rangle$? (Note: These expectation values can also be computed in the usual way (as in problem 3 below). However, once you have obtained the probabilities of different outcomes, these quantities can be obtained in a very simple way.

3. Suppose that an electron spin is in the spin state given below

$$\psi = N \begin{pmatrix} 5 \\ 3 + 4i \end{pmatrix}$$

where $N$ is a normalization constant.

(a) Find a value for $N$ so that $\psi$ is normalized.

(b) Find the expectation values of $\langle S_x \rangle$, $\langle S_y \rangle$ and $\langle S_z \rangle$. What is $\langle \vec{S} \rangle$?

(c) What are the expectation values of $\langle S_x^2 \rangle$, $\langle S_y^2 \rangle$, $\langle S_z^2 \rangle$ and $\langle S^2 \rangle$?

(d) Use the result in part (b) to find the probabilities of obtaining spin-up ($p_\uparrow$) and spin-down ($p_\downarrow$) outcomes when

i. $S_x$ measured,
ii. $S_y$ is measured and
iii. $S_z$ is measured.

Hint: Remember the relation between the expectation values of observables and the probabilities of different outcomes (which you have used in 2(g). Also remember the probability sum rule. Using these two relations, you can compute $p_\uparrow$ and $p_\downarrow$.

4. Let $\hat{n}$ be a unit vector and $\psi$ is a state such that $\langle \sigma_n \rangle = 1$.

(a) What is $\langle S_n \rangle$?

(b) What is the uncertainty $\Delta \sigma_n$? What is $\Delta S_n$? (Hint: What is $\sigma_n^2$?)

(c) Show (argue) that $\psi$ is an eigenstate of $\sigma_n$ (and of $S_n$). What is the eigenvalue?

(d) Let $\phi$ be an arbitrary spin state

$$\phi = \begin{pmatrix} a \\ b \end{pmatrix}$$

where the amplitudes satisfy the relation $|a|^2 + |b|^2 = 1$. Compute $\langle \vec{\sigma} \rangle$ in terms of the amplitudes and show that $\langle \vec{\sigma} \rangle$ is a unit vector.

(e) Let’s call the unit vector in part (d) as $\hat{m}$. Using the reasoning above, argue that $\phi$ is the spin-up state along $\hat{m}$ (i.e., $\phi$ is an eigenstate of $\sigma_m$).

(f) Consider the state $\psi$ in problem 2. This state is a spin-up state along some direction. Which direction is it?

5. Consider the spin-up along x state $\alpha_x$ (which you have computed in problem 2). Verify the following relations (which we had computed in the angular momentum quantization
chapter an argued that this state is “physically invariant” under rotations around $x$-axis.

\[
\langle S_z \rangle = m\hbar , \\
\langle S_y \rangle = \langle S_z \rangle = 0 , \\
\langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{2}(j(j + 1) - m^2) , \\
\langle (S_y \cos \alpha + S_z \sin \alpha)^2 \rangle = \frac{\hbar^2}{2}(j(j + 1) - m^2) \text{ for any } \alpha .
\]

What are the values of $j$ and $m$?

6. Consider the state

\[
\psi = \frac{1}{\sqrt{2}} \left[ \begin{array}{c}
1 \\
e^{i\phi}
\end{array} \right] .
\]

(a) Find $\langle \vec{\sigma} \rangle$.

(b) Let $D(z, \alpha)$ be the rotation operator for the rotation around $z$ axis by angle $\alpha$. Let

\[
\psi' = D(z, \alpha)\psi .
\]

Compute $\langle \vec{\sigma} \rangle$ in the state $\psi'$. Do you get what you have expected?

(c) The case is very subtle for the angle $\alpha = 2\pi$ (i.e., $360^\circ$ rotation). Show that for

\[
\psi' = D(z, 2\pi)\psi
\]

we have $\psi' \neq \psi$, but these two states have the same $\langle \vec{\sigma} \rangle$. Argue what this means.

(d) Show that $D(z, 4\pi) = I$. 

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