PHYS 508 (2015-1)
Homework III
Due date: Dec. 7, 2015, Monday.

1. Every attractive potential in 1D has a bound state. Consider a particle in 1D moving under the effect of an attractive potential \( V(x) \). We will assume that \( V(x) \to 0 \) as \( x \to \pm \infty \) and \( V(x) \leq 0 \) everywhere. This is what we mean by an "attractive potential". Use the variational approach to show that the exact ground-state energy \( E_{gs} \) of the particle is negative, \( E_{gs} < 0 \) (in other words, the particle is bound, it cannot escape to infinity). Hint: Choose a Gaussian trial wavefunction, \( \psi(x) = N e^{-\beta x^2/2} \), and investigate the behavior of \( \langle H \rangle_\psi \) around \( \beta \to 0 \) limit.

Remark: This fabulous result is also true in 2D, but not in 3D!

2. Consider a hydrogen atom inside a uniform electric field along the z direction, \( \vec{E} = E \hat{z} \). The Hamiltonian is

\[
H = \frac{p^2}{2m} - \frac{e^2}{r} + eEz
\]

and we treat this problem in degenerate perturbation theory. In this problem, you are going to find how the levels will split for \( n = 3 \) levels of the hydrogen atom.

(a) First, let us look at the matrix elements of \( z \). Using the properties of the spherical harmonics, show that

\[
\langle \psi_{n\ell m} | z | \psi_{n'\ell' m'} \rangle = 0 \text{ if } m \neq m'.
\]

Also, argue that \( \langle \psi_{n\ell m} | z | \psi_{n'\ell' m'} \rangle = 0 \) if \( \ell - \ell' \) is even.

Note: By Wigner-Eckart theorem, the matrix element \( \langle \psi_{n\ell m} | z | \psi_{n'\ell' m'} \rangle \) can be non-zero only if \( m = m' \) and \( \ell - \ell' = \pm 1 \).

(b) Discuss the following symmetries for the perturbed Hamiltonian: Parity, time-reversal, rotation (space and spin), and the "hidden-symmetry" associated with Runge-Lenz vector. Which ones are broken? What are the good quantum numbers? (For example: Is \( \ell \) a good quantum number? What about \( m, or m_s \)?)

Note: You can answer all these questions by symmetry arguments. Do not try to compute a commutator!

(c) Now, discuss qualitatively the effect of perturbation on the \( n = 3 \) levels with spin included (i.e., all 18 states). You can safely assume that all matrix elements that can be non-zero are definitely non-zero. But, do not evaluate the matrix elements and do not diagonalize any matrices. Answer the following questions:

* The \( n = 3 \) levels split to how many levels by the perturbation?
* What are the degeneracies of each split level?
* What are the quantum numbers associated with each level?

Note: We have answered some of these questions in our treatment of Stark effect in class. You will have an opportunity to go over these. You may want to display the energy levels as energy vs \( E \) curves.
3. Hydrogen atom with spin-orbit coupling is

\[ H = \frac{p^2}{2m} - \frac{e^2}{r} + \frac{A}{r^3} \vec{L} \cdot \vec{S}, \]

where \( A \) is a positive constant. In this problem, you will look at the effect of the spin-orbit coupling on \( n = 3 \) and \( n = 4 \) levels of the hydrogen atom. As in problem 2, you only need to do this qualitatively. Do not compute any integrals. (But, keep in mind that \( \langle r^{-3} \rangle \) depends on \( \ell \) quantum number and decreases with increasing \( \ell \).)

(a) First, discuss the symmetries. Which ones are broken? What are the good quantum numbers? Which states get mixed by perturbation?

(b) Discuss the splitting of \( n = 3 \) levels. What are the degeneracies of each split level? Which quantum numbers are associated with each?

(c) Do the same for \( n = 4 \) levels.

4. Consider the energy levels of a particle under gravitational force. Let \( x \)-axis show the upward direction and suppose that \( x = 0 \) plane is the ground. We will assume that the particles cannot penetrate below the ground, so that we will assume that there is a hard-wall boundary at \( x = 0 \), i.e., potential is infinite there. We can also consider this as a 1D problem. If the mass of the particle is \( m \), the potential energy function can be written as

\[ V(x) = \begin{cases} \infty & \text{for } x \leq 0, \\ mgx & \text{for } x > 0. \end{cases} \]

(a) Use WKB approximation and the closely related quantization condition (i.e., old quantum theory) to find the energy of the \( n \)th level.

(b) Another approach is to use variation calculation. As the wavefunction should vanish at the origin, a trial wavefunction of the form

\[ \psi_{\text{trial}}(x) = Nxe^{-\beta x^2/2} \]

would be appropriate. Find the best value of \( \beta \) and obtain an approximation for the ground state energy \( E_0 \).

Note: \[ \int_0^{\infty} x^{2n} e^{-\beta x^2} dx = \left( -\frac{\partial}{\partial \beta} \right)^n \int_0^{\infty} e^{-\beta x^2} dx, \]

\[ \int_0^{\infty} x^{2n+1} e^{-\beta x^2} dx \text{ can be converted into a } \Gamma \text{ integral.} \]

(c) Get a numerical estimate for the typical height of the particle above the ground for an electron in the ground state. (This would be either the turning point for WKB, or \( \beta^{-1/2} \) for the variational approach.)